Algebraic Geometry
Part III Michaelmas 2016-2017

Alexandre Daoud

May 27, 2017

Contents

1 Basic Definitions

1.1 Sheaves and Stalks

Definition 1.1.1. Let X be a topological space, $\mathbf{Op}(X)$ the poset of open sets of X considered as a category and $\mathcal{C}$ a category. We define a **presheaf** of $\mathcal{C}$-objects on X, denoted $\mathcal{F}$, to be a contravariant functor $\mathcal{F} : \mathbf{Op}(X) \to \mathcal{C}$. Given an open set $U \subseteq X$, we refer to the elements of $\mathcal{F}(U)$ as the **sections** of U. Moreover, given an inclusion of open sets $V \subseteq U$ we say that $\mathcal{F}(U \subseteq V) = F(V) \to F(U)$ is the **restriction** of U to V where $s \in \mathcal{F}(U)$ is mapped to $s|_V \in \mathcal{F}(V)$.

Finally, we define a **sheaf** on X to be a presheaf $\mathcal{F}$ such that if

$$U = \bigcup_i U_i$$

for some open sets $U_i \subseteq X$ and if $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Example 1.1.2. Let X be a topological space. Then the functor $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{AbGrp}$ given by

$$\mathcal{F}(U) = \{ \text{continuous functions } U \to \mathbb{R} \}$$

is a sheaf.

Example 1.1.3. From now on, $\mathcal{C}$ will either be $\mathbf{AbGrp}, \mathbf{Ring}$ or $\mathbf{Mod_R}$ for some commutative ring R. Moreover, *sheaf* shall be synonymous with *sheaf of $\mathcal{C}$-objects*.

Definition 1.1.4. Let $(I, \leq)$ be a directed poset. Suppose for each $i \in I$ we have an abelian group A_i and for each pair $i \leq j$ we have a map $\varphi_{ij} : A_i \to A_j$ with $\varphi_{ii} = \mathrm{id}_{A_i}$ such that whenever $i \leq j \leq k$, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. Then we say that (A_i, φ_{ij}) is a **directed system** of abelian groups.

Moreover, consider pairs (A_i, a_i) with $a_i \in A_i$. Define an equivalence relation on these pairs where $(A_i, a_i) \sim (A_j, a_j)$ if and only if there exists a $k \geq i, j$ such that $\varphi_{ik}(a_i) = \varphi_{jk}(a_j)$. Denoting the equivalence class of (A_i, a_i) under $\sim$ as $[A_i, a_i]$, we may define a group operation on the set of all such equivalence classes as follows:

$$[A_i, a_i] + [A_j, a_j] = [A_k, \varphi_{ik}(a_i) + \varphi_{jk}(a_j)]$$

for any $k \geq i, j$. We call this group the **direct limit** of the direct system (A_i, φ_{ij}) and we denote it by $\varinjlim_{i \in I} A_i$.

Definition 1.1.5. Let X be a topological space, $\mathcal{F}$ a presheaf of abelian groups on X and $x \in X$. Consider the directed poset $(I, \subseteq)$ consisting of open sets containing x, ordered by inclusion. Then $\mathcal{F}(U_i)$, together with the restriction homomorphisms, define a direct system. We define the **stalk** of $\mathcal{F}$ at x by

$$\mathcal{F}_x = \varinjlim_{U_i \in I} \mathcal{F}(U_i)$$

Definition 1.1.6. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ presheaves of abelian group on X. We define a **morphism** of presheaves to be a natural transformation $\varphi : F \to G$. In other words, φ is given by a collection of group homomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that if $V \subseteq U$ then the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow |_V & & \downarrow |_V \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

is commutative. Moreover, we say that φ is an **isomorphism** of presheaves if it has an inverse. We denote by $\mathbf{Sh}(X)$ the category of all sheaves on X together with their morphisms.

Remark. Given a morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ and a point $x \in X$ there is a natural homomorphism of stalks

$$\begin{aligned} \varphi_x : \mathcal{F}_x &\to \mathcal{G}_x \\ (U, s) &\mapsto (U, \varphi_U(s)) \end{aligned}$$

Theorem 1.1.7. *Let X be a topological space and $\mathcal{F}$ a presheaf of abelian groups on X. Then there exists a sheaf $\mathcal{F}^+$ and a morphism $\alpha : \mathcal{F} \to \mathcal{F}^+$ such that, given any sheaf $\mathcal{G}$ and morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, φ factors through $\mathcal{F}^+$ uniquely:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{F}^+ \\ & \underset{\varphi}{\searrow} & \downarrow \\ & & \mathcal{G} \end{array}$$

*for some morphism of sheaves $\mathcal{F}^+ \to \mathcal{G}$. We shall refer to $\mathcal{F}^+$ as the **sheaf associated** to $\mathcal{F}$ or the **sheafification** of $\mathcal{F}$.*

Proof. Fix an open set $U \subseteq X$ and let $\{ U_i \}_{i \in I}$ be an open cover for some indexing set I. We claim that

$$\mathcal{F}^+(U) = \left\{ s : U \to \dot{\bigcup_{x \in U}} \mathcal{F}_x \;\middle|\; \begin{array}{c} \forall\, x \in U, s(x) \in \mathcal{F}_x \\ \exists\, x \in W \subseteq U \text{ open}, t \in \mathcal{F}(W) \text{ s.t } s(y) = [W, t]\ \forall\, y \in W \end{array} \right\}$$

defines the desired sheaf along with the natural restriction morphisms. This clearly defines a presheaf so it thus suffices to show that $\mathcal{F}^+$ satisfies the sheaf axiom. Let $s_i \in \mathcal{F}^+(U_i)$ be sections such that for all $i, j \in I$ we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Define a function

$$\begin{aligned} s : U &\to \dot{\bigcup_{x \in U}} \mathcal{F}_x \\ y &\mapsto s_i(y) \end{aligned}$$

for some i such that $y \in U_i$. Then s is well-defined since the sections s_i all agree on overlaps. Now, given any $x \in U$, we clearly have $s(x) \in \mathcal{F}_x$ since $s(x) = s_i(x)$ for any $i \in I$ such that $U_i \ni x$. Furthermore, for each s_i, there exists an open neighbourhood $x \in W_i \subseteq U$ and a section $t \in \mathcal{F}(W)$ such that for all $y \in W_i$ we have $s_i(y) = [W, t]$. Clearly, we can take any of these W_i and the same will apply for s whence $s \in \mathcal{F}^+(U)$. Lastly, we must show that such an s is unique. To this end, suppose there exists a $t \in \mathcal{F}^+(U)$ such that their restrictions $s_i, t_i \in \mathcal{F}^+(U_i)$ agree. Then for all $x \in U$, there exists a $U_i \ni x$ such that $s_i(x) = t_i(x)$ and so $s(x) = t(x)$. Since this holds for all $x \in U$, we must have that $s = t$. We have thus shown that $\mathcal{F}^+$ is indeed a sheaf.

Now, given $s \in \mathcal{F}(U)$, define $\alpha : \mathcal{F} \to \mathcal{F}^+$ by setting $\alpha_U(s)$ to be the function mapping $x \in U$ to $[U, s]$. This is easily seen to be a morphism of presheaves as it is compatible with the natural restriction morphisms.

To see that φ factors uniquely through $\mathcal{F}^+$, we must construct a unique morphism of sheaves $\psi : \mathcal{F}^+ \to \mathcal{G}$. To this end, fix $s \in \mathcal{F}^+(U)$ and for each U_i in the open cover, choose $s_i \in \mathcal{F}(U_i)$ such that $\alpha_{U_i}(s_i) = s|_{U_i}$. Now set $t_i = \varphi(s_i)$. Since φ is a morphism of presheaves, it follows that $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. Since $\mathcal{G}$ is a sheaf, there exists a unique $t \in \mathcal{G}(U)$ such that $t|_{U_i} = t_i$. We must therefore have that $\psi_U(s) = t$ and we are done. $\square$

Remark. Let X be a topological space and $\mathcal{F}$ a presheaf. For all $x \in X$, we have a homorphism of groups

$$\alpha_x : \mathcal{F}_x \to \mathcal{F}_x^+$$

This is infact an isomorphism since the sections of $\mathcal{F}^+$ are locally just sections of $\mathcal{F}$.

Example 1.1.8. Let $X = \{ a, b \}$ be a topological space where the open sets are $\varnothing, X, U = \{ a \}$ and $V = \{ b \}$. Define a presheaf of abelian groups on X by setting

$$\mathcal{F}(\varnothing) = 0, \quad \mathcal{F}(X) = \mathbb{Z}, \quad \mathcal{F}(U) = 0, \quad \mathcal{F}(V) = 0$$

with the natural restriction homomorphisms. We first calculate the stalks of $\mathcal{F}$. Recall that the stalk at a is given by

$$\mathcal{F}_a = \frac{\{ (A, s) \mid A \ni a, s \in \mathcal{F}(A) \}}{\sim}$$

where $\sim$ is the equivalence relation given by $(U, s) \sim (V, t)$ if and only if there exists an open $a \in W \subseteq U \cap V$ such that $s|_W = t|_W$. We have that

$$\{ (A, s) \mid A \ni a, s \in \mathcal{F}(A) \} = \{ (U, 0), \dots, (X, -1), (X, 0), (X, 1), \dots \}$$

Clearly the elements of this set are all equivalent so we have $\mathcal{F}_a = 0$. Similarly, we find that $\mathcal{F}_b = 0$. It then follows that all sections of $\mathcal{F}^+$ are necessarily 0.

Example 1.1.9. Let $X = \{ a, b \}$ be a topological space where the open sets are $\varnothing, X, U = \{ a \}$ and $V = \{ b \}$. Define a presheaf of abelian groups on X by setting

$$\mathcal{F}(\varnothing) = 0, \quad \mathcal{F}(X) = 0, \quad \mathcal{F}(U) = \mathbb{Z}, \quad \mathcal{F}(V) = \mathbb{Z}$$

We again calculate the stalks of this presheaf. The set to consider in the direct limit for $\mathcal{F}_a$ is

$$\{ (A, s) \mid A \in a, s \in \mathcal{F}(A) \} = \{ (X, 0), \dots, (U, -1), (U, 0), (U, 1), \dots \}$$

Clearly the only equivalent elements are $(X, 0)$ and $(U, 0)$ so $\mathcal{F}_a = \mathbb{Z}$. Similarly, we have $\mathcal{F}_b = \mathbb{Z}$. By the definition of the sheafification, we then have that $\mathcal{F}^+(U) = \mathcal{F}^+(V) = \mathbb{Z}$ and $\mathcal{F}^+(X) = \mathbb{Z} \oplus \mathbb{Z}$.

Definition 1.1.10. Let X be a topological space and $\varphi : F \to G$ a morphism of presheaves. We define the **presheaf kernel** of φ, denoted $\ker \varphi^{\text{pre}}$ by

$$(\ker \varphi^{\text{pre}})(U) = \ker (\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))$$

Similarly, we define the **presheaf image** of φ, denoted $\operatorname{im} \varphi^{\text{pre}}$ by

$$(\operatorname{im} \varphi^{\text{pre}})^+(U) = \operatorname{im}(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))$$

Furthermore, if $\mathcal{F}$ and $\mathcal{G}$ are also sheaves then we also have the **sheaf kernel**, denoted $\ker \varphi$, defined in the same way and the **sheaf image**, defined by $\operatorname{im} \varphi = (\operatorname{im} \varphi^{\text{pre}})^+$.

Finally, we say that φ is injective if $\ker \varphi = 0$ and **surjective** if $\operatorname{im} \varphi = G$.

Proposition 1.1.11. *Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves. Then $\ker \varphi^{\text{pre}}$ and $\operatorname{im} \varphi^{\text{pre}}$ are presheaves of abelian groups. If, in addition, $\mathcal{F}$ and $\mathcal{G}$ are sheaves then $\ker \varphi$ is also a sheaf.*

Proof. Since the kernel of any homomorphisms of abelian groups is again an abelian group, $\ker \varphi^{\text{pre}}$ indeed assigns an abelian group to each open set $U \subseteq X$. Furthermore, since the mapping between the empty sets is vacuously 0, we have that $(\ker \varphi^{\text{pre}})(\varnothing) = 0$. Finally, the restriction homomorphisms are made evident in the following diagram[1]:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\;|_V\;} & \mathcal{F}(V) \\ \big\downarrow{\scriptstyle |_{\ker \varphi_U}} & & \big\downarrow{\scriptstyle |_{\ker \varphi_V}} \\ (\ker \varphi^{\text{pre}})(U) & \longrightarrow & (\ker \varphi^{\text{pre}})(V) \end{array}$$

A similar argument also shows that $\operatorname{im} \varphi^{\text{pre}}$ is a presheaf. To show that $\ker \varphi$ is a sheaf, assume that we are given an open set $U \subseteq X$ and an open cover $U = \bigcup_{i \in I} U_i$ for some indexing set I. Suppose that $s_i \in (\ker \varphi)(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j. We need to show that there exists a unique $s \in (\ker \varphi)(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. Since $(\ker \varphi)(U) \subseteq \mathcal{F}(U)$ and $\mathcal{F}$ is a sheaf, it follows that the sections local s_i glue together to give a global section $s \in \mathcal{F}(U)$. We claim that such an s is the desired global section. To this end, we have that $\varphi(s_i) = 0$ for all $i \in I$. Since $\mathcal{G}$ is a sheaf, these local sections must glue together to give a global section $\varphi(s) = 0$. Hence $s \in (\ker \varphi)(U)$. The uniqueness of such an s follows immediately from the fact that $\mathcal{F}$ is a sheaf. □

Example 1.1.12. Let $X = \{ a, b \}$ be a topological space where the open sets are $\varnothing, X, U = \{ a \}$ and $V = \{ b \}$. Define a sheaf on X by setting

$$\mathcal{F}(\varnothing) = 0, \quad \mathcal{F}(X) = \mathbb{Z} \oplus \mathbb{Z}, \quad \mathcal{F}(U) = \mathbb{Z}, \quad \mathcal{F}(V) = \mathbb{Z}$$

Define the sheaf $\mathcal{G}$ on X by setting

$$\mathcal{G}(\varnothing) = 0, \quad \mathcal{F}(X) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad \mathcal{F}(U) = \mathbb{Z}/2\mathbb{Z}, \quad \mathcal{F}(V) = \mathbb{Z}/2\mathbb{Z}$$

Furthermore, define a morphism of sheaves between $\varphi : \mathcal{F} \to \mathcal{G}$ by setting

$$\begin{aligned} \varphi_X : \mathcal{F}(X) &\to \mathcal{G}(X) \\ (m, n) &\mapsto (\overline{m}, \overline{n}) \end{aligned}$$

$$\begin{aligned} \varphi_U : \mathcal{F}(U) &\to \mathcal{G}(U) \\ m &\mapsto \overline{m} \end{aligned}$$

$$\begin{aligned} \varphi_V : \mathcal{F}(V) &\to \mathcal{G}(V) \\ n &\mapsto \overline{n} \end{aligned}$$

Then $(\ker \varphi)(X) = 2\mathbb{Z} \times 2\mathbb{Z}$ and $(\ker \varphi)(U) = 2\mathbb{Z} = (\ker \varphi)U$ and $\operatorname{im} \varphi = \mathcal{G}$.

Theorem 1.1.13. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of shaves on a topological space X. Then*

1. *φ is injective if and only if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$.*

2. *φ is surjective if and only if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$.*

3. *φ is an isomorphism if and only if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$.*

[1]while slightly overloading notation for the restriction maps

Proof. Part 1: First suppose that φ is injective, fix some $x \in X$ and choose an equivalence class $[U, s] \in \mathcal{F}_x$. Then $0 = \varphi_x([U, s]) = [U, \varphi_U(s)]$ implies that there exists an open $W \ni x$ with $W \subseteq U$ such that $\varphi_U(s)|_W = 0$. This in turn implies that $\varphi_W(s|_W) = 0$. Now φ is injective by hypothesis so $s|_W = 0$. Hence $0 = [W, s] = [U, s]$ as desired.

Now suppose that φ_x is injective for all $x \in X$. Given an open set $U \subseteq X$, assume that $\varphi_U(s) = 0$ with $s \in \mathcal{F}(U)$. We then have that

$$0 = [U, 0] = [U, \varphi_U(s)] = \varphi_x([U, s])$$

Since φ_x is injective, we thus have that $[U, s] = 0$. This implies that there exists some $W \ni x$ open with $W \subseteq U$ and $s|_W = 0$. Since this applies to all $x \in X$ and since $\mathcal{F}$ is a sheaf, it follows that $s = 0$.

Part 2: Assume that φ is surjective, in other words, $(\operatorname{im} \varphi^{\text{pre}})^+ = \mathcal{G}$. Then the homomorphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is just

$$\varphi_x : \mathcal{F}_x \to (\operatorname{im} \varphi^{\text{pre}})_x^+ \cong \operatorname{im} \varphi_x^{\text{pre}}$$

which is trivially surjective.

Now suppose that φ_x is surjective for all $x \in X$. We want to show that for all open neighbourhoods $U \subseteq X$, the group homomorphism

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is surjective. To this end, fix an open $U \subseteq X$ and let $t \in \mathcal{G}(U)$. We need to show that there exists an $s \in \mathcal{F}(U)$ such that $\varphi_U(s) = t$. By hypothesis, given x, we have that for all $[W, b] \in \mathcal{G}_x$, there exists a $[V, a] \in \mathcal{F}_x$ such that

$$\varphi_x([V, a]) = [W, b]$$

In particular, there exists an $s \in \mathcal{F}(U)$ and an open neighbourhood $x \in V \subseteq U$ such that $\varphi_x([V, s]) = [U, t]$. But the left hand side of this equation is equal to $[V, \varphi_U(s_x)]$. By the definition of a stalk, this is equivalent to there existing an open neighbourhood $x \in W \subseteq V$ such that $\varphi_U(s)|_W = t$. In other words, sections of $\mathcal{G}$ are just locally the images of sections of $\mathcal{F}$. Passing to the sheafification, we then have that $\operatorname{im} \mathcal{F} = \mathcal{G}$ as desired.

Part 3: First suppose that φ is an isomorphism. Then it is injective and surjective and by Parts 1 and 2, φ_x is an isomorphism for each $x \in X$.

Conversely, suppose that each φ_x is an isomorphism for all $x \in X$. By Parts 1 and 2, φ is injective and surjective. Let $\mathcal{H} = \operatorname{im} \varphi^{\text{pre}}$. Since φ is injective, $\mathcal{F}(U)$ is isomorphic to $\mathcal{H}(U)$ for all open sets $U \subseteq X$. In particular, $\mathcal{H}$ is a sheaf isomorphic to $\mathcal{F}$. Since φ is surjective, $\mathcal{H}^+ = \mathcal{G}$. Since $\mathcal{H}$ is a sheaf, $\mathcal{H} = \mathcal{G}$. Hence φ is an isomorphism.

□

Definition 1.1.14. Let X be a topological space. We define a **complex** of sheaves to be a sequence

$$\cdots \longrightarrow \mathcal{F}_{-1} \xrightarrow{\varphi_0} \mathcal{F}_0 \xrightarrow{\varphi_1} \mathcal{F}_1 \xrightarrow{\varphi_2} \mathcal{F}_2 \xrightarrow{\varphi_2} \cdots$$

such that $\operatorname{im} \varphi_i \subseteq \ker \varphi_{i+1}$ for all i. We say that this complex is an **exact sequence** if we have $\operatorname{im} \varphi_i = \ker \varphi_{i+1}$ for all i. Furthermore, an exact sequence of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is called a **short** exact sequence.

Example 1.1.15. Let X be a topological space and A an abelian group. Define a presheaf $\mathcal{F}$ by setting $\mathcal{F}(U) = A$ for all non-empty open sets $U \subseteq X$. We call $\mathcal{F}^+$ the **constant sheaf** associated to A. Also define the sheaf $\mathcal{G}$ by

$$\mathcal{G}(U) = \{ \text{continuous functions } U \to A \}$$

where A is equipped with the discrete topology. Define a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ by sending $s \in \mathcal{F}(U)$ to the constant function

$$\begin{aligned} f_s : U &\to A \\ u &\mapsto s \end{aligned}$$

Then φ induces an isomorphism of sheaves $\varphi : \mathcal{F}^+ \to \mathcal{G}$. This follows from showing the stalks of the two sheaves are isomorphic. Indeed, to show that $\varphi_x : \mathcal{F}^+_x \to \mathcal{G}_x$ is an injective, suppose that $\varphi_x([U, s]) = 0$. By definition, we have that $[U, \varphi_U(s)] = 0$. This just means that, locally, $\varphi_U(s)$ is the zero function whence $s = 0$ and so $[U, s] = 0$.

For surjectivity, choose $[V, t] \in \mathcal{G}_x$. We need to exhibit a $[U, s] \in \mathcal{F}_x$ such that $\varphi_x([U, s]) = [V, t]$. By definition, t is a continuous function $t : V \to A$ so set $s = t(x)$ and $U = t^{-1}(\{ s \})$. We claim that $[U, s]$ is the desired element of $\mathcal{F}_x$. We have that $\varphi_x([U, s]) = [U, \varphi_U(s)] = [U, f_s]$. Then $[U, f_s] \sim [V, t]$ if and only if there exists an open neighbourhood $x \in W$ such that $W \subseteq U \cap V$ and $f_s|_W = t|_W$. However, we may simply take $W = U$ and we are done.

Definition 1.1.16. Let X and Y be a topological space and $f : X \to Y$ a continuous mapping. If $\mathcal{F}$ is a presheaf on X, we define the **direct image** of $\mathcal{F}$ with respect to f, denoted f_*, to be the assignment

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$$

giving rise to a presheaf on Y.

Proposition 1.1.17. *Let X and Y be topological spaces, $f : X \to Y$ a continuous mapping and $\mathcal{F}$ a sheaf on X. Then $(f_*\mathcal{F})$ is a sheaf on Y.*

Proof. The direct image is clearly a presheaf on Y with the natural restriction morphisms. To show that it is a sheaf, let $V \subseteq Y$ be an open neighbourhood and $\{ V_i \}_{i \in I}$ an open cover of V where I is some indexing set. Choose $t_i \in (f_*\mathcal{F})(V_i)$ such that $t_i|_{V_i \cap V_j} = t_j|_{V_i \cap V_j}$ for all i, j. Each t_i is in $\mathcal{F}(f^{-1}V_i)$ and satisfies $t_i|_{f^{-1}V_i \cap f^{-1}V_j} = t_j|_{f^{-1}V_i \cap f^{-1}V_j}$ for all i, j. Since $\mathcal{F}$ is a sheaf, there exists a unique $t \in f^{-1}V$ such that $t|_{f^{-1}V_i} = t_i$ for all i. Hence there exists a $t \in (f_*\mathcal{F})(V)$ such that $t|_{V_i} = t_i$ for all i. Thus, the direct image is a sheaf. □

Example 1.1.18. Let X be a topological space, $x \in X$ and A an abelian group. Define a sheaf on X by setting

$$\mathcal{F}(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

where $U \subseteq X$ is an open set. This is referred to as the **skyscraper** sheaf associated to A at x. Let $Z = \{ x \}$ and define the inclusion map

$$i : Z \hookrightarrow X$$

Let $\mathcal{G}$ be the constant sheaf on Z associated to A. Then $\mathcal{F} = i_*\mathcal{G}$.

1.2 Results from Commutative Algebra

Henceforth, all rings are assumed to be commutative.

Definition 1.2.1. Let R be a ring. We say that R is **local** if it has a unique maximal ideal.

Definition 1.2.2. Let R and S be local rings with maximal ideals $\mathfrak{m}_R$ and $\mathfrak{m}_S$. A homomorphism of rings $\alpha : R \to S$ is said to be **local** if $\alpha(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.

Definition 1.2.3. Let R be a ring and $I \lhd R$ an ideal. We define the **radical** of I, denoted $\sqrt{I}$ to be the set

$$\sqrt{I} = \{ r \in R \mid r^n \in I, n \in \mathbb{N} \}$$

Proposition 1.2.4. *Let R be a ring and $I \lhd R$ an ideal. Then*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$$

where the intersection is taken over all prime ideals $\mathfrak{p}$ contained in I.

Proof. Omitted. □

Proposition 1.2.5. *Let K be algebraically closed and $I \lhd K[t_1, \ldots, t_n]$ a maximal ideal. Then $I = (t_1 - a_1, \ldots, t_n - a_n)$ for some $a_i \in K$.*

Proof. Omitted. □

Definition 1.2.6. Let R be a ring and $S \subseteq R$ a subset. We say that S is **multiplicatively closed** if $1_R \in S$ and $s, t \in S$ implies that $st \in S$.

Definition 1.2.7. Let R be a ring and $S \subseteq R$ a multiplicatively closed subset. Consider the set

$$\left\{ \frac{r}{s} \,\middle|\, r \in R, s \in S \right\}$$

of formal fractions. Define an equivalence relation on this set with $a/s \sim b/s'$ if and only if there exists $s'' \in S$ such that $s''(as' - bs) = 0$. We define

$$S^{-1}A = \left\{ \frac{r}{s} \,\middle|\, r \in R, s \in S \right\} / \sim$$

to be the **ring of fractions** of R with respect to S with ring operations given by

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$
$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

Example 1.2.8. Let $R = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{ 0 \}$. Then $S^{-1}R = \mathbb{Q}$.

Remark. There is a natural inclusion homomorphism

$$\alpha : R \hookrightarrow S^{-1}R$$
$$r \mapsto \frac{r}{1}$$

Proposition 1.2.9. *Let R be a ring and $I \lhd R$ an ideal. Then*

$$S^{-1}I = \left\{ \frac{r}{s} \in S^{-1}R \;\middle|\; r \in I \right\}$$

is an ideal of $S^{-1}R$. Moreover, any ideal of $S^{-1}R$ is of this form.

Proof. Fix an ideal of $I \lhd R$. We must show that $(S^{-1}I, +)$ is a subgroup of $(S^{-1}R, +)$ and that for all $S^{-1}I$ absorbs multiplication by elements of $S^{-1}R$.

$S^{-1}I$ clearly contains the additive identity of $S^{-1}R$ since I contains the additive identity of R. Fix $a/b, c/d \in S^{-1}I$ where $a, c \in I$ and $b, d \in S$. Then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Now, S is multiplicatively closed so $bd \in S$. Furthermore, $ad + bc \in I$ so indeed $a/b + c/d \in S^{-1}I$. Clearly, all elements of $S^{-1}I$ have additive inverses so $(S^{-1}I, +)$ is indeed a subgroup of $(S^{-1}R, I)$. To prove that S^{-1} absorbs multiplication by elements of $S^{-1}R$, choose $a/b \in S^{-1}I$ and $c/d \in S^{-1}R$. Then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

As before, $bd \in S$ and $ac \in I$ so the product of the two fractions is again in $S^{-1}I$ whence it is an ideal of $S^{-1}R$.

To show that any ideal of the ring of fractions is of this form, choose an ideal $J \lhd S^{-1}R$. Let I be the set consisting of all numerators of fractions in J. We claim that I is an ideal of R, it would then immediately follow that $J = S^{-1}I$.

I clearly contains the additive identity of R since J is an ideal of $S^{-1}R$. Furthermore, given $a, b \in I$, $a + b \in I$ since $a/1 + b/1 = (a + b)/1 \in J$. I also clearly contains additive inverses and so $(I, +)$ is a subgroup of $(R, +)$. Now let $i \in I$ and $r \in R$. Choose any fraction in J with i as its numerator, say $i/j \in J$. Then $i/j \cdot r/1 = ir/j \in J$ and so $ir \in I$ whence I is an ideal. □

Proposition 1.2.10. *Let R be a ring and $S \subseteq R$ a multiplicatively closed subset. Then there is a one-to-one inclusion preserving correspondence*

$$\left\{ \begin{matrix} \text{prime } \mathfrak{p} \lhd R \\ \mathfrak{p} \cap S = \varnothing \end{matrix} \right\} \longleftrightarrow \left\{ \text{prime } \mathfrak{p} \lhd S^{-1}R \right\}$$

$$\mathfrak{p} \longleftrightarrow S^{-1}\mathfrak{p}$$

Proof. We must check that the correspondence is well-defined and the two mappings are mutually inverse. To this end, fix a prime ideal $\mathfrak{p} \lhd R$ such that $\mathfrak{p} \cap S = \varnothing$ and let $a/b \cdot c/d \in S^{-1}\mathfrak{p}$. We need to show that either $a/b \in S^{-1}\mathfrak{p}$ or $c/d \in S^{-1}\mathfrak{p}$. Choose u, v such that $ab/cd = u/v$. Then there exists $z \in S$ such that $z(abv - cdu) = 0$. It then follows that $zabv \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, one of z, a, b or v must be in $\mathfrak{p}$. But $\mathfrak{p} \cap S = \varnothing$ so it cannot be z or v. Hence either a or b is in $\mathfrak{p}$ whence either a/b or $c/d \in S^{-1}\mathfrak{p}$.

Conversely, suppose that $\mathfrak{q} \lhd S^{-1}R$ is prime. We need to show that the ideal $\mathfrak{p}$ consisting of all numerators in $\mathfrak{q}$ is prime. To this end, let $ab \in \mathfrak{p}$. Choose a fraction in $\mathfrak{q}$ with ab as its numerator, say ab/cd. By definition this is equal to $a/b \cdot c/d \in \mathfrak{q}$. But $\mathfrak{q}$ is prime so either $a/c \in \mathfrak{q}$ or $b/d \in \mathfrak{q}$ whence either a or b is in $\mathfrak{p}$. Thus the maps are well defined and do map prime ideals to prime ideals.

We must now check that the maps are mutually inverse. Label the forward mapping φ and the backwards map ψ. First let $\mathfrak{p} \lhd R$ be prime. We want to show that $\psi(\varphi(\mathfrak{p})) = \mathfrak{p}$. □

Definition 1.2.11. Let R be a ring and $\mathfrak{p} \lhd R$ a prime ideal. Define a multiplicative subset $S = R \setminus \mathfrak{p}$. We call the ring of fractions $S^{-1}R$ the **localisation** of R at $\mathfrak{p}$ and denote it $R_\mathfrak{p}$.

Proposition 1.2.12. *Let R be a ring and $\mathfrak{p} \lhd R$ prime. Then $R_\mathfrak{p}$ is a local ring with unique maximal ideal given by $\mathfrak{p}_\mathfrak{p} := S^{-1}\mathfrak{p}$.*

Proof. Let $\mathfrak{m}$ be an ideal not contained in $\mathfrak{p}_\mathfrak{p}$. Choose a fraction $a/b \in \mathfrak{m}$. Then both a and b are contained in $R \backslash \mathfrak{p}$. By definition of the ring of fractions, this implies that the fraction b/a is an element of $R_\mathfrak{p}$. Hence $1_{R_\mathfrak{p}} = a/b \cdot b/a \in \mathfrak{m}$ whence $\mathfrak{m} = R_\mathfrak{p}$. □

Remark. Let R be a ring and let $S = \{1, b, b^2, \dots\}$ be a multiplicatively closed power set for some $b \in R$. We shall write $R_b = S^{-1}R$.

Moreover, note that all these definitions can be generalised to arbitrary modules over a commutative ring. More precisely, if R is a commutative ring, M an R-module and S^{-1} a multiplicative set in R then $S^{-1}M$ is an $S^{-1}R$-module. Moreover, if $M \to N$ is an R-module homomorphism, we then have an induced homomorphism $S^{-1}M \to S^{-1}N$ of $S^{-1}R$-modules. In fact, $S^{-1}(\cdot)$ is an exact functor $\mathbf{Mod_R} \to \mathbf{Mod_{S^{-1}R}}$.

Definition 1.2.13. Let R be a ring and M, N R-modules. Let L denote the free R-module generated by elements of $M \times N$. Let E be the sub-R-module of L generated by elements of the form

1. $(m + m', n) - (m, n) - (m', n')$
2. $(m, n + n') - (m, n) - (m, n')$
3. $(rm, n) - r(m, n)$
4. $(m, rn) - r(m, n)$

where $m, m' \in M$, $n, n' \in N$ and $r \in R$. We define the **tensor product** of M and N over R to be

$$M \otimes_R N = L/E$$

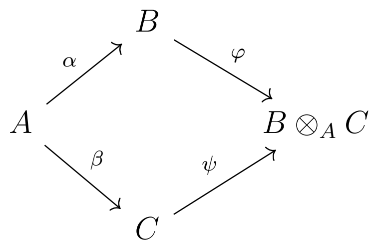
and we write $m \otimes n$ for the equivalence class of (m, n).

Proposition 1.2.14. *Let R be a ring and N, M and P R-modules. Then*

1. *If $M \times N \to P$ is an R-bilinear map then there exists a unique homomorphism of modules $M \otimes_R N \to P$.*
2. *$R \otimes_R M \cong M$.*
3. *$M \otimes_R N = N \otimes_R M$.*
4. *$(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$.*
5. *$M \otimes_R (N \oplus P) \cong (M \otimes_R N) \oplus (M \otimes_R P)$.*
6. *If $S \subseteq R$ is multiplicatively closed we have $S^{-1}M \cong S^{-1}R \otimes_R M$.*
7. *If $I \lhd R$ we have $R/I \otimes_R M \cong M/IM$.*

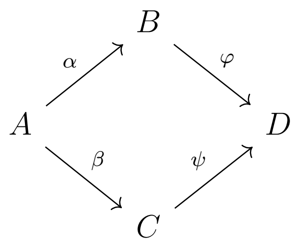
Proof. Ommitted. □

Remark. Let A, B, C and D be rings and $\alpha : A \to B, \beta : A \to C$. Then we have a commutative diagram

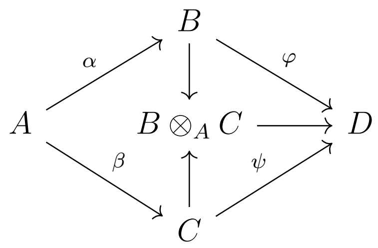


where φ sends b to $b \otimes 1$ and ψ sends c to $1 \otimes c$.

Proposition 1.2.15. *Let A, B, C and D be rings and suppose we have a commutative diagram*



Then there exists a unique homomorphism of A-modules $B \otimes_A C \to D$ extending the diagram to a commutative diagram



1.3 Spectrum of a Ring

Definition 1.3.1. Let R be a ring. We define the **spectrum** of R, denoted $\operatorname{Spec} R$, to be the set of all prime ideals of R. Moreover, given any ideal $I \lhd R$, we define $V(I) = \{ \mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p} \}$.

Lemma 1.3.2. *Let R be a ring. Then*

1. *For all $I, J \lhd R$ we have $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.*
2. *For all families of ideals $I_\alpha \lhd R$ we have $V\left(\sum_\alpha I_\alpha\right) = \bigcap_\alpha V(I_\alpha)$.*
3. *For all $I, J \lhd R$ we have $V(I) \subseteq V(J)$ if and only if $\sqrt{I} \supseteq \sqrt{J}$.*

Proof.

<u>Part 1:</u> We have that

$$\mathfrak{p} \in V(IJ) \iff IJ \subseteq \mathfrak{p} \iff I \subseteq \mathfrak{p} \text{ or } J \subseteq \mathfrak{p} \iff \mathfrak{p} \in V(I) \text{ or } \mathfrak{p} \in V(J)$$

A similar argument applies to $V(I \cap J)$.

Part 2: We have that

$$\mathfrak{p} \in V\left(\sum_{\alpha} I_\alpha\right) \iff \sum_{I_\alpha} I_\alpha \subseteq \mathfrak{p} \iff I_\alpha \subseteq \mathfrak{p}\ \forall \alpha \iff \mathfrak{p} \in \bigcap_{\alpha} V(I_\alpha)$$

Part 3: By Proposition 1.2.4, we have that $\sqrt{I} = \bigcap V(I)$ and $\sqrt{J} = \bigcap V(J)$. The statement then follows immediately.

□

Definition 1.3.3. Let R be a ring. We define the **Zariski** topology on $X = \operatorname{Spec} R$ by declaring the closed sets of X to be the $V(I)$. Moreover, we define the **structure sheaf** of X, denoted by $\mathcal{O}_X$, to be the sheaf of rings

$$\mathcal{O}_X(U) = \left\{ s : U \to \dot{\bigcup_{\mathfrak{p}\in U}} R_\mathfrak{p} \;\middle|\; \begin{array}{c} \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in R_\mathfrak{p} \\ \exists \mathfrak{p} \in W \subseteq U \text{ open s.t } \forall \mathfrak{q} \in W, s(\mathfrak{q}) = \dfrac{a}{b} \in R_\mathfrak{q} \end{array} \right\}$$

Proposition 1.3.4. *Let R be a ring and $X = \operatorname{Spec} R$. Then $\mathcal{O}_X$ is indeed a sheaf.*

Proof. $\mathcal{O}_X$ is clearly a presheaf with the natural restriction homomorphisms. We just need to check the sheaf condition. To this end, let $U \subseteq X$ be an open set and $U = \bigcup_i U_i$ be an open cover of U. Suppose that $s_i \in \mathcal{O}_X(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j. Define a function

$$s : U \to \dot{\bigcup_{\mathfrak{p}\in U}} R_\mathfrak{p}$$
$$\mathfrak{p} \mapsto s_i(\mathfrak{p})$$

where i is chosen whenever $\mathfrak{p} \in U_i$. Then this function is well-defined as the s_i agree on overlaps. We claim that s is the desired section in the sheaf condition. It's restriction to U_i is clearly just s_i so we must have that $s \in \mathcal{O}_X(U)$ and that such an s is unique. □

Proposition 1.3.5. *Let R be a ring and $X = \operatorname{Spec} R$. Then*

$$\{\, D(b) = X \setminus V((b)) \mid b \in R \,\}$$

is a basis for the Zariski Topology on X.

Proof. It suffices to show that the $D(b)$ are open in X and that any given any open set $U \subseteq X$ and a prime $x \in U$, there exists a $b \in R$ such that $x \in D(b) \subseteq U$.

Now, fix $b \in R$, it is immediate that $D(r)$ is open as, by definition, $D(b) = X \setminus V((r))$ and $X \setminus D(b) = V((b))$ is closed.

Next, fix an open neighbourhood $U \subseteq X$ and a prime $\mathfrak{p} \in U$. By definition, $U = X \setminus V(I)$ for some ideal $I \subseteq R$. Moreover, $\mathfrak{p}$ does not contain I. Choose any non-zero element $b \in I$. Then $\mathfrak{p}$ does not contain (b) so that $\mathfrak{p} \notin V((b))$ whence $\mathfrak{p} \in X \setminus V((b)) = D(b)$. By construction, $D(b) \subseteq U$ thereby proving the proposition. □

Theorem 1.3.6. *Let R be a ring and $X = \operatorname{Spec} R$. Then*

1. *$(\mathcal{O}_X)_\mathfrak{p} \cong R_\mathfrak{p}$ as local rings for all $\mathfrak{p} \in X$.*

2. $\mathcal{O}_X(D(b)) \cong R_b$ *for all* $b \in R$.

3. $\mathcal{O}_X(X) \cong R$.

Proof.

Part 1: Define a ring homomorphism

$$\begin{aligned} f : (\mathcal{O}_X)_\mathfrak{p} &\to R_\mathfrak{p} \\ [U, s] &\mapsto s(\mathfrak{p}) \end{aligned}$$

We claim that f is the desired local isomorphism. We must first check that f is well-defined. Suppose that $[U, s] = [V, t]$. Then, by the definition of a stalk, there exists an open neighbourhood $\mathfrak{p} \in W \subseteq U \cap V$ such that $s|_W = t|_W$. It then follows that $s(\mathfrak{p}) = t(\mathfrak{p})$.

We now show that f is injective. Assume that $f([U, s]) = s(\mathfrak{p}) = 0$. By definition, s is given by some fraction a/b on some open neighbourhood $\mathfrak{p} \in W \subseteq U$. So $s(\mathfrak{p}) = 0$ implies that there exists some $c \notin \mathfrak{p}$ such that $ca = 0$. It then follows that we have $a/b = 0$ in all local rings $R_\mathfrak{q}$ such that $b, c \notin \mathfrak{q}$. Equivalently, $\mathfrak{q} \in D(b) \cap D(c)$. Then s is 0 on the neighbourhood of $\mathfrak{p}$ given by $D(b) \cap D(c) \cap W$ whence $[U, s] = 0$ and f is injective.

We next show that f is surjective. Choose a fraction $a/b \in R_\mathfrak{p}$. Let $U = D(b)$ and $s \in \mathcal{O}_X(U)$ be given by a/b. Then, clearly, $f([U, s]) = a/b$ as desired.

Finally, we must show that this in fact a local isomorphism. It suffices to show that the set

$$\mathfrak{m} = \{\, [U, s] \mid f([U, s]) = s(\mathfrak{p}) \in \mathfrak{p}_\mathfrak{p} \,\}$$

is the unique maximal ideal of $(\mathcal{O}_X)_\mathfrak{p}$. Let $I \lhd (\mathcal{O}_X)_\mathfrak{p}$ be an ideal not contained in $\mathfrak{m}$. We need to show that all elements of I are invertible. To this end, fix $[U, s] \in (\mathcal{O}_X)_\mathfrak{p}$. Then $f([U, s]) = s(\mathfrak{p}) \notin \mathfrak{p}_\mathfrak{p}$ and is thus invertible in $R_\mathfrak{p}$. Let $s(\mathfrak{p})^{-1}$ denote its inverse in $R_\mathfrak{p}$. Then since f is a ring isomorphism, $f^{-1}(s(\mathfrak{p}))$ is an inverse for $[U, s]$ in $(\mathcal{O}_X)_\mathfrak{p}$ and we are done.

Part 2: Define a ring homomorphism

$$\begin{aligned} g : R_b &\to \mathcal{O}_X(D(b)) \\ \frac{a}{b^n} &\mapsto \left(\text{sections defined by } \frac{a}{b^n}\right) \end{aligned}$$

We claim that g is an isomorphism. We first show that it is injective. To this end, suppose that $g(a/b^n) = 0$. Then for all $\mathfrak{p} \in D(b)$, $a/b^n = 0$ in $R_\mathfrak{p}$. For such a $\mathfrak{p}$ we have that there exists $c_\mathfrak{p} \notin \mathfrak{p}$ such that $c_\mathfrak{p} a = 0$. Define $I = (c_\mathfrak{p})_{\mathfrak{p} \in D(b)}$. Then $D(b) \cap V(I) = \varnothing$. Indeed

$$\mathfrak{p} \in D(b) \implies c_\mathfrak{p} \notin \mathfrak{p} \implies I \not\subseteq \mathfrak{p} \implies \mathfrak{p} \notin V(I)$$

Hence $V(I) \subseteq V((b))$ whence $\sqrt{I} \supseteq \sqrt{(b)}$. By definition of the radical, we thus have $b^r \in I$ for some $r \in \mathbb{N}$ so $b^r = \sum_i d_i c_{\mathfrak{p}_i}$. Multiplying by a we get

$$ab^r = \sum_i d_i a c_{\mathfrak{p}_i} = 0$$

And so $a/b^n = 0$ in A_b.

We must now show that g is surjective. To this end, choose a section $s \in \mathcal{O}_X(D(b))$ and let $\{\, U_i \,\}_{i \in I}$ be an open cover of $D(b)$. Suppose that $s|_{U_i}$ is given by some a_i/e_i. We may assume that each $U_i = D(d_i)$ for some $d_i \in R$. From this we observe that $D(d_i) \subseteq D(e_i)$

and so $\sqrt{(d_i)} \subseteq \sqrt{(e_i)}$. By the definition of the radical, we have $d_i^{n_i} = c_i e_i$ for some $n_i \in \mathbb{N}$ and $c_i \in R$. We may replace

$$\frac{a_i}{e_i} = \frac{c_i a_i}{c_i e_i} = \frac{c_i a_i}{d_i^{n_i}}$$

Noting that $D(d_i) = D(d_i^{n_i})$ for all n_i, we may assume that $U_i = D(e_i)$. So then $D(b) = \bigcup_i D(e_i)$ whence

$$V((b)) = \bigcap_i V((e_i)) = V\left(\sum_i (e_i)\right)$$

Again applying the radical identity we have $\sqrt{(b)} = \sqrt{\sum(e_i)}$. This implies that $b^n = \sum_{\text{finite}} l_j e_j$ for some $l_j \in R$. Going back through the identities, we may then adjust the indexing so we have a finite union

$$D(b) = \bigcup_{\text{finite}} D(e_i)$$

Now by hypothesis, a_i/e_i and a_k/e_k define the same section on $D(e_i) \cap D(e_j) = D(e_i e_k)$. By Part 1, the homomorphism $R_{e_i e_k} \to \mathcal{O}_X(D(e_i e_k))$ is injective and so $a_i/e_i = a_k/e_k$ in $R_{e_i e_k}$. By definition of the ring of fractions, there exists an $n' \in \mathbb{N}$ such that

$$(e_i e_k)^{n'}(a_i e_k - a_k e_i) = e_k^{n'+1} e_i^{n'} a_i - e_i^{n'+1} e_k^{n'} a_k = 0$$

for all i, k. By equivalence, we may then assume that $a_i e_k = a_k e_i$. From this it follows that

$$e_k \left(\sum_i l_i a_i\right) = \sum_i l_i a_i e_k = \sum_i l_i a_k e_i = a_k \sum_i l_i e_i = a_k b^n$$

and so

$$\frac{a_k}{e_k} = \sum_i \frac{l_i a_i}{b^n}$$

Hence s is given by $\sum_i l_i a_i / b^n$ on $D(b)$ and therefore g is surjective.

Part 3: This follows directly from Part 2 by taking $b = 1$.

□

1.4 Ringed Spaces

Definition 1.4.1. A **ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings called the **structure sheaf** of X. We say that $(X, \mathcal{O}_X)$ is a **locally** ringed space if $(\mathcal{O}_X)_\mathfrak{p}$ are local rings for all $\mathfrak{p} \in X$.

Definition 1.4.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A **morphism** $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of

1. a continuous map $f : X \to Y$.

2. a morphism of sheaves $\varphi : \mathcal{O}_Y \to f_* \mathcal{O}_X$.

Furthermore, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces then φ is a morphism of locally ringed spaces if the induced homomorphism

$$\begin{aligned}(\mathcal{O}_Y)_{\mathfrak{q}} &\to (\mathcal{O}_X)_{\mathfrak{p}} \\ [V, t] &\mapsto [f^{-1}V, s]\end{aligned}$$

is a local homomorphism for $\mathfrak{q} = f(\mathfrak{p})$. Finally, an **isomorphism** of (locally) ringed spaces is a morphism which has an inverse.

Theorem 1.4.3. *Let R and S be rings, $(X = \operatorname{Spec}(R), \mathcal{O}_X), (Y = \operatorname{Spec}(S), \mathcal{O}_Y)$ ringed spaces and $\alpha : R \to S$ a homomorphism of rings. Then*

1. *$(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces.*
2. *α induces a morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed spaces.*
3. *Any morphism of locally ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is induced by some ring homomorphism $\alpha : R \to S$.*

Proof.

Part 1: This follows immediately from Theorem 1.3.6.

Part 2: We first define $f : Y \to X$ by setting $f(\mathfrak{p}) = \alpha^{-1}(\mathfrak{p})$ for $\mathfrak{p} \in Y$. It is easy to see that f is continuous. Indeed, given a closed set $V(I)$, its inverse image under f is simply $V((\alpha I))$ which is again closed.

We now define φ. Recall that given $\mathfrak{p} \in Y$ with $\mathfrak{q} = f(\mathfrak{p})$ we have a local homomorphism

$$\begin{aligned}\alpha_{\mathfrak{p}} : R_{\mathfrak{q}} &\to S_{\mathfrak{p}} \\ \frac{a}{b} &\mapsto \frac{\alpha(a)}{\alpha(b)}\end{aligned}$$

Now, choose $s \in \mathcal{O}_X(U)$ for some open $U \subseteq X$. Recall that s is a function

$$s : U \to \dot{\bigcup_{\mathfrak{q} \in U}} R_{\mathfrak{q}}$$

Define a section $t \in \mathcal{O}_X(f^{-1}U)$ by

$$\begin{aligned}t : f^{-1}U &\to \dot{\bigcup_{\mathfrak{p} \in f^{-1}U}} S_{\mathfrak{p}} \\ \mathfrak{p} &\mapsto \alpha_{\mathfrak{p}}(s(f(\mathfrak{p})))\end{aligned}$$

If s is locally given by a/b then t is locally given by $\alpha(a)/\alpha(b)$. This gives a morphism of sheaves

$$(f, \varphi) : (Y, \mathcal{O}_Y(U)) \to (X, \mathcal{O}_X(U))$$

as desired. Now, the homomorphism induced on stalks by φ is simply $\alpha_{\mathfrak{p}}$ and so this is indeed a morphism of locally ringed spaces.

Part 3:

Suppose $(f, \varphi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of locally ringed spaces. By Part 3 of Theorem 1.3.6, applying (f, φ) to the global section X yields a homomorphism of rings $\alpha : R \to S$. We claim that (f, φ) is induced by α.

To show this, fix $\mathfrak{p} \in Y$ and set $\mathfrak{q} = f(\mathfrak{p})$. Consider the commutative diagram

$$\begin{array}{ccc} R = \mathcal{O}_X(X) & \xrightarrow{\alpha} & \mathcal{O}_Y(Y) = S \\ \downarrow \beta & & \downarrow \gamma \\ R_{\mathfrak{q}} = (\mathcal{O}_X)_{\mathfrak{q}} & \xrightarrow{\alpha_{\mathfrak{p}}} & (\mathcal{O}_Y)_{\mathfrak{p}} = S_{\mathfrak{p}} \end{array}$$

From this we may read off

$$\mathfrak{q} = \beta^{-1}(\mathfrak{q}_{\mathfrak{q}}) = \beta^{-1}(\alpha_{\mathfrak{p}}^{-1}(\mathfrak{p}_{\mathfrak{p}})) = \alpha^{-1}(\gamma^{-1}(\mathfrak{p}_{\mathfrak{p}})) = \alpha^{-1}(\mathfrak{p})$$

whence $f = \alpha^{-1}$. To see that φ is also induced by α, let $U \subseteq X$ be an open set and $\mathfrak{p} \in U$ with $\mathfrak{q} = f(\mathfrak{p})$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{\varphi_U} & \mathcal{O}_Y(f^{-1}(U)) \\ \downarrow & & \downarrow \\ (\mathcal{O}_X)_{\mathfrak{q}} & \xrightarrow{\alpha_{\mathfrak{p}}} & (\mathcal{O}_Y)_{\mathfrak{p}} \end{array}$$

Fix a section $s \in \mathcal{O}_X(U)$. Then this section is determined by all the values $s(\mathfrak{p}) \in \mathcal{O}_Y(f^{-1}U)$. The commutative diagram then makes it clear that φ is determined by α. □

2 Schemes

2.1 Definitions

Definition 2.1.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. We say that $(X, \mathcal{O}_X)$ is an **affine scheme** if it isomorphic to $(X = \operatorname{Spec}(R), \mathcal{O}_X)$ for some ring R. We say that $(X, \mathcal{O}_X)$ is a **scheme** if for all $x \in X$ there exists an open neighbourhood $x \in U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. A **morphism** of schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a morphism between them as locally ringed spaces. We denote by $\mathbf{Sch}(X)$ the category of schemes over X and their morphisms.

Remark. Henceforth, by an abuse of notation, an (affine) scheme $(X, \mathcal{O}_X)$ will be written simply as X. The stalks $(\mathcal{O}_X)_x$ shall be written as $\mathcal{O}_{X,x}$ or simply $\mathcal{O}_x$.

Example 2.1.2. Let K be a field. Then $X = \operatorname{Spec}(K)$ is a scheme consisting of a single point (the only prime ideal of a field is the zero ideal). Furthermore, if L/K is a field extension then $Y = \operatorname{Spec}(L) \to X = \operatorname{Spec}(K)$ is a morphism of schemes.

Example 2.1.3. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$. Then $\operatorname{Spec}(R) = \{0, \mathfrak{m}\}$. The stalks are given by $\mathcal{O}_0 = R_0 = \operatorname{Frac}(R)$ and $\mathcal{O}_{\mathfrak{m}} = R_{\mathfrak{m}}$.

Example 2.1.4. Let $X = \operatorname{Spec}(\mathbb{Z}) = \{0, (2), (3), (5), \dots\}$. The stalk at $x = 0$ is simply $\mathbb{Q}$. If $x = (p)$ for some prime number p then $\mathcal{O}_x = \mathbb{Z}_{(p)}$. Note that if $\mathfrak{m}_p$ is the maximal ideal of $\mathbb{Z}_{(p)}$ then $\mathbb{Z}_{(p)}/\mathfrak{m}_p \cong \mathbb{F}_p$.

Furthermore, if R is any ring then the characteristic ring homomorphism

$$\begin{aligned} \mathbb{Z} &\to R \\ n &\mapsto n \cdot 1_R \end{aligned}$$

induces a morphism of schemes $\operatorname{Spec}(R) \to \operatorname{Spec}(\mathbb{Z})$.

Definition 2.1.5. Let R be a ring. We define **affine n-space** over R, denoted $\mathbb{A}^n_R$, to be

$$\mathbb{A}^n_R = \operatorname{Spec}(R[t_1, \ldots, t_n])$$

Example 2.1.6 (Classical Algebraic Geometry)**.** Let K be an algebraically closed field and $I \lhd K[t_1, \ldots, t_n]$ an ideal. Since $K[t_1, \ldots, t_n]$ is Noetherian, we have that $I = (f_1, \ldots, f_r)$ for some $f_i \in K[t_1, \ldots, t_n]$. Consider the set

$$S = \{\, (a_1, \ldots, a_n) \mid a_i \in K, f_j(a_1, \ldots, a_n) = 0 \;\forall\, j \,\}$$

Then there exists a one-to-one correspondence between S and the set of maximal ideals in $K[t_1, \ldots, t_n]$ containing I (in other words, ideals of the form $(t_1 - a_1, \ldots, t_n - a_n)$). classical algebraic geometry studies S whereas modern algebraic geometry studies $\operatorname{Spec} K[t_1, \ldots, t_n]/I$.

Definition 2.1.7. Let X be a scheme. We say that X is **irreducible** if for all non-empty open sets $U, V \subseteq X$ we have $U \cap V \neq \varnothing$. Equivalently, if $X = Y \cup Z$ for Y and Z closed then either $X = Y$ or $X = Z$.

Definition 2.1.8. Let R be a ring. We say that R is **reduced** if $\operatorname{nil}(R) = 0$. Furthermore, if X is a scheme, we say that X is **reduced** if for all open sets $U \subseteq X$, $\mathcal{O}_X(U)$ is reduced.

Definition 2.1.9. Let X be a scheme. We say that X is **integral** if for all open sets $U \subseteq X$, $\mathcal{O}_X(U)$ is an integral domain.

Proposition 2.1.10. *Let $X = \operatorname{Spec}(R)$ be an affine scheme for some ring R. Then*

1. *X is irreducible if and only if $\operatorname{nil}(R)$ is a prime ideal of R.*
2. *X is reduced if and only if R is reduced.*
3. *X is irreducible and reduced if and only if R is an integral domain.*

Proof.

Part 1: We have that X is irreducible if and only if $X = V(I) \cup V(J)$ implies that $X = V(I)$ or $X = V(J)$. Recall that $V(I) \cup V(J) = V(IJ)$ and that $\operatorname{nil}(R)$ is the intersection of all prime ideals in a ring. From this we see that X is irreducible if and only $IJ \subseteq \operatorname{nil}(R)$ implies that $I \subseteq \operatorname{nil}(R)$ or $J \subseteq \operatorname{nil}(R)$. But this is exactly what it means for $\operatorname{nil}(R)$ to be prime.

Part 2: The forward direction is just by definition so assume that R is reduced. Let $s \in \mathcal{O}_X(U)$ be nilpotent. Then for all $x \in U$, the image of s in $\mathcal{O}_x = R_x$ is nilpotent. By hypothesis, R_x is reduced so $s = 0$ in R_x for all $x \in U$. Since $\mathcal{O}_X$ is a sheaf, it follows that $s = 0$ in $\mathcal{O}_X(U)$ whence $\mathcal{O}_X(U)$ is reduced.

Part 3: We have that X is irreducible and reduced if and only if $\operatorname{nil}(R)$ is prime and $\operatorname{nil}(R) = 0$. But this is equivalent to R being and integral domain.

□

Theorem 2.1.11. *Let X be a scheme. Then X is integral if and only if it is irreducible and reduced.*

Proof. First suppose that X is integral. Then clearly X is reduced. Now assume that there exists open sets $U, V \subseteq X$ such that $U \cap V = \varnothing$. Then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ since $\mathcal{O}_X$ is a sheaf. But the direct sum of two non-zero rings can never be an integral domain which is a contradiction.

Conversely, suppose that X is irreducible and reduced. We first claim that for all open sets $U \subseteq X$ and $x \in U$, there exists an open affine neighbourhood $x \in W \subseteq U$.

By the definition of a scheme, there exists an open affine $V = \operatorname{Spec}(R) \subseteq X$ such that $x \in V$. Then there exists $b \in R$ such that $x \in D(b) \subseteq U \cap V$. Now, as schemes, we have that $D(b) \cong \operatorname{Spec}(R_b)$ so the claim is proved.

Now suppose that $s, t \in \mathcal{O}_X(U)$ such that $st = 0$ with $s \neq 0$. We need to show that $t = 0$. By the claim, we can cover U by open affine sets $U = \bigcup V_i$ where $V_i = \operatorname{Spec}(R_i)$ for some ring R_i. Then for some i, $s|_{V_i} \neq 0$. Since X is irreducible and reduced, so is V_i. Proposition 2.1.10 then implies that R_i is an integral domain and so

$$st|_{V_i} = s|_{V_i} \cdot t|_{V_i} = 0$$

implies that $t|_{V_i} = 0$. We claim that in fact $t|_{V_j} = 0$ for all j.

Now, X is irreducible whence $V_i \cap V_j \neq \varnothing$ for all j. Since $t|_{V_i \cap V_j} = 0$, we must then have that $t = 0$ in $\mathcal{O}_x$ for all $x \in V_i \cap V_j$. Note that $\mathcal{O}_x \cong (R_j)_x$ and the natural inclusion

$$\begin{aligned} R_j &\to (R_j)_x \\ a &\mapsto \frac{a}{1} \end{aligned}$$

is injective. Since the image of $t|_{V_j}$ is 0 under this map, it follows that $t|_{V_j} = 0$ for all j. But $\mathcal{O}_U$ is a sheaf whence $t = 0$. Hence $\mathcal{O}_X(U)$ is an integral domain and X is integral. □

Definition 2.1.12. Let X be a scheme. We say that $\eta \in X$ is **generic** if $\overline{\{\eta\}} = X$.

Proposition 2.1.13. *Let X be an integral scheme. Then X has a unique generic point.*

Proof. Let U be any affine open set $U = \operatorname{Spec}(R)$ for some ring R. We claim that $\eta = 0 \triangleleft R$ is a generic point of U. Let $I \triangleleft R$ be an ideal. Then $V(I)$ clearly never contains the zero ideal unless $I = 0$. Since $V(0) = \operatorname{Spec}(R)$, it follows that every non-empty open subset of U contains η which is exactly what it means for η to be dense in U. Now suppose that η' is any other generic point of U. Then, by definition, $\eta' \in V$ for all non-empty open subsets of U. Then the only I such that $\eta' \in V(I)$ is $I = 0$. Hence η' is a minimal prime ideal of R. Since X is integral, so is U when viewed as a scheme whence R is an integral domain. Since 0 is the unique minimal prime ideal of an integral domain, we must have that $\eta' = 0 = \eta$ and so U has a unique generic point.

Now, X is integral and, in particular, it is irreducible. This is equivalent to every non-empty open subset of X being dense in X. Since $\eta = 0$ is dense in all non-empty open subsets U when viewed as a scheme, η is thus also dense in X and we are done. □

Proposition 2.1.14. *Let X be an integral scheme and η its unique generic point. Then $\mathcal{O}_\eta$ is a field called the* ***function field*** *of X and denoted $K(X)$.*

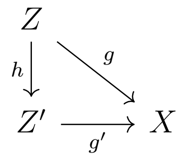
Proof. Let $U \subseteq X$ be any affine open set where $U = \operatorname{Spec}(R)$. Then $\mathcal{O}_\eta = (\mathcal{O}_X)_\eta = (\mathcal{O}_U)_\eta = R_{(0)} = \operatorname{Frac}(R)$. □

Definition 2.1.15. Let X and Y be schemes and $f : Y \to X$ a morphism. We say that f is an **open immersion** if $U := f(Y)$ is open in X and f induces an isomorphism of locally ringed spaces $(Y, \mathcal{O}_Y) \to (U, \mathcal{O}_X|_U)$. An **open subscheme** of X is any open immersion of some scheme Y to X.

Definition 2.1.16. Let X and Z be schemes. A **closed immersion** is a morphism of schemes $g : Z \to X$ such that

1. $g(Z)$ is closed in X.
2. g induces a homeomorphism $Z \to g(Z)$.
3. $\mathcal{O}_X \to g_* \mathcal{O}_Z$ is a surjection.

A **closed subscheme** of X is any closed immersion from some scheme Z into X up to the following equivalence relation. Two closed immersions $g : Z \to X$ and $g' : Z' \to X$ define the same closed subscheme if there exists an isomorphism $h : Z \to Z'$ such that the diagram



commutes.

Example 2.1.17. Let $X = \mathrm{Spec}(R)$ for some ring R and $I \lhd R$ an ideal. Then $R \to R/I$ gives a closed immersion $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$.

2.2 Schemes Associated to Graded Rings

Definition 2.2.1. Let S be a ring. We say that S is **graded** if there exist a collection of rings $\{ S_d \}_{d\in\mathbb{N}}$ such that $S = \bigoplus_{d\in\mathbb{N}} S_d$ and $S_d S_c \subseteq S_{d+c}$. If $s_d \in S_d$ then we say that s_d is **homogeneous of degree d**.

Example 2.2.2. $\mathbb{C}[t_1, \ldots, t_n]$ is a graded ring.

Definition 2.2.3. Let $S = \bigoplus_{d\in\mathbb{N}} S_n$ be a graded ring and $I \lhd S$ an ideal. We say that I is a **homogeneous** ideal if

$$I = \bigoplus_{d\in\mathbb{N}} I \cap S_d$$

Proposition 2.2.4. *Let $S = \bigoplus_{d\in\mathbb{N}}$ be a graded ring and $I, J \lhd S$ homogeneous ideals. Then $I + J, IJ, I \cap J$ and $\sqrt{I}$ are all homogeneous ideals.*

Proof. We have that

$$I + J = \left(\bigoplus_{d\in\mathbb{N}} I \cap S_d \right) + \left(\bigoplus_{d\in\mathbb{N}} J \cap S_d \right) = \bigoplus_{d\in\mathbb{N}} (I + J) \cap S_d$$

A similar argument shows that IJ and $I \cap J$ are also homogeneous ideals.

To show that $\sqrt{I}$ is homogeneous, choose $s \in \sqrt{I}$. Then $s^n \in I$ for some $n \in \mathbb{N}$. Without loss of generality, we may suppose that s^n is homogeneous of degree d with $s^n \in I_d$. Since I is homogeneous, we must have that $s \in I_{d/n}$. The elements of $\sqrt{I}$ are thus homogeneous and we are done. □

Proposition 2.2.5. *Let $S = \bigoplus_{d\in\mathbb{N}} S_d$ be a graded ring and $\mathfrak{p} \lhd S$ a homogeneous ideal. If for all homogeneous ideals $I, J \lhd S$ we have that $IJ \subseteq \mathfrak{p}$ implies $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$ then $\mathfrak{p}$ is prime.*

Proof. Let a and b be elements (not necessarily homogeneous) such that $ab \in \mathfrak{p}$. Suppose that neither a nor b is in $\mathfrak{p}$. Let $a = \sum_i a_i$ and $b = \sum_j b_j$ be their homogeneous expansions. Since $a \notin \mathfrak{p}$ and the terms in the expansion are eventually 0, there exists a maximum d such that $a_d \notin \mathfrak{p}$. Similarly, there exists a maximum e such that $b_e \notin \mathfrak{p}$.

Since $ab \in \mathfrak{p}$, all of its components are as well. The $(d+e)^{th}$ component of ab is given by $\sum_{i+j=d+e} a_i b_j$. Each pair (i,j) except (d,e) must satisfy either $i > d$ or $j > e$. The maximality of d and e then imply that each $a_i b_j \in \mathfrak{p}$. This then implies that $a_i b_j \in \mathfrak{p}$. By hypothesis, either a_i or b_j is in $\mathfrak{p}$ which is a contradiction. □

Definition 2.2.6. Let S be a graded ring and $\mathfrak{p} \triangleleft S$ a homogeneous prime ideal. We define the **homogeneous localisation** of S at $\mathfrak{p}$ by

$$S_{(\mathfrak{p})} = \left\{ \frac{a}{b} \in S_{\mathfrak{p}} \;\middle|\; a, b \text{ are homogeneous and have the same degree} \right\}$$

Similarly, given a homogeneous element of non-zero degree $b \in S$ we define

$$S_{(b)} = \left\{ \frac{a}{b^r} \in S_b \;\middle|\; a, b^r \text{ are homogeneous and have the same degree} \right\}$$

Definition 2.2.7. Let $S = \bigoplus_{d \in \mathbb{N}} S_d$ be a graded ring and $S_+ = \bigoplus_{d>0} S_d$. We define the **homogeneous spectrum** of S to be the set

$$\mathrm{Proj}(S) = \{ \mathfrak{p} \triangleleft S \mid \mathfrak{p} \text{ is homogeneous and } S_+ \not\subseteq \mathfrak{p} \}$$

Furthermore, for all $I \triangleleft S$, define

$$V_+(S) = \{ \mathfrak{p} \in \mathrm{Proj}(S) \mid I \subseteq \mathfrak{p} \}$$

Lemma 2.2.8. *Let S be a graded ring. Then*

1. *For all homogeneous ideals $I, J \triangleleft S$ we have $V_+(IJ) = V_+(I \cap J) = V_+(I) \cup V_+(J)$.*
2. *For any family of homogeneous ideals I_α of S we have $V_+(\sum_\alpha I_\alpha) = \cap_\alpha V_+(I_\alpha)$.*

Proof. Follows a similar argument to the affine case. □

Definition 2.2.9. Let S be a graded ring. We can define a topology on $X = \mathrm{Proj}(S)$ called the **Zariski** topology by taking the closed sets to be the $V_+(I)$ for all $I \triangleleft S$. Moreover, we define the **structure sheaf** of X, denoted $\mathcal{O}_X$ to be the sheaf of rings

$$\mathcal{O}_X(U) = \left\{ s : U \to \dot{\bigcup_{\mathfrak{p} \in U}} S_{(\mathfrak{p})} \;\middle|\; \begin{array}{c} \forall\, \mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})} \\ \exists \text{ open } \mathfrak{p} \in W \subseteq U \text{ such that } \forall\, \mathfrak{q} \in W, \\ s(\mathfrak{q}) = \dfrac{a}{b} \in S_{(\mathfrak{q})} \text{ where } a, b \in S \text{ are homogeneous of the same degree} \end{array} \right\}$$

Proposition 2.2.10. *Let S be a graded ring and $X = \mathrm{Proj}(S)$. Then*

$$\{ D_+(b) = X \setminus V_+((b)) \mid b \in S \textit{ homogeneous} \}$$

is a basis for the Zariski topology on X.

Proof. This is proven in a similar way to the affine case. □

Theorem 2.2.11. *Let $S = \bigoplus_{d \in \mathbb{N}} S_d$ be a graded ring and $X = \mathrm{Proj}(S)$. Then*

1. *$(\mathcal{O}_X)_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$ for all $\mathfrak{p} \in X$.*

2. *For all homogeneous $b \in S_+$ there exists a natural isomorphism of locally ringed spaces between $D_+(b)$ and $\operatorname{Spec}(S_{(b)})$.*

3. *$(X, \mathcal{O}_X)$ is a scheme.*

Proof.

Part 1: Similar argument to the affine case.

Part 2: First denote $U_b := D_+(b)$ and $Y := \operatorname{Spec}(S_{(b)})$. We shall construct an isomorphism of locally ringed spaces

$$(f, \varphi) : (U_b, \mathcal{O}_X|_{U_b}) \to (Y, \mathcal{O}_Y)$$

Note that we have natural homomorphisms of rings $S \to S_b$ and $S_{(b)} \hookrightarrow S_b$. We use these to define f as follows:

$$\begin{aligned} f : U_b &\to Y \\ \mathfrak{p} &\mapsto \mathfrak{p}_b \cap S_{(b)} \end{aligned}$$

We first show that f is injective. Suppose that $f(\mathfrak{p}) = f(\mathfrak{q})$ for some $\mathfrak{p}, \mathfrak{q} \in U_b$. We need to show that $\mathfrak{p} = \mathfrak{q}$. To this end, fix $x \in \mathfrak{p}$. Let $x = \sum_i x_i$ be its homogeneous expansion. Since $\mathfrak{q}$ is homogeneous, it suffices to show that each $x_i \in \mathfrak{q}$. By hypothesis, we have that

$$\mathfrak{p}_b \cap S_{(b)} = \mathfrak{q}_b \cap S_{(b)}$$

Now, we can always find $n, r \in \mathbb{N}$ such that $\deg(x_i^n) = \deg(b^r)$ so for such n and r, we have that $x_i^n/b^r \in \mathfrak{p}_b \cap S_{(b)}$. But then $x_i^n/b^r \in \mathfrak{q}_b \cap S_{(b)}$. This means that $x_i^n \in \mathfrak{q}$. Since $\mathfrak{q}$ is prime, we thus have that $x_i \in \mathfrak{q}$ and so $\mathfrak{p} \subseteq \mathfrak{q}$. A similar argument gives us the reverse inclusion whence f is injective.

We next show that f is surjective. Fix $\mathfrak{q} \in Y = \operatorname{Spec}(S_{(b)})$. We need to exhibit $\mathfrak{p} \in U_b = D_+(b)$ such that $f(\mathfrak{p}) = \mathfrak{q}$. Define

$$I_m = \left\{ a \in S_m \;\middle|\; \frac{a^{\deg(b)}}{b^m} \in \mathfrak{q} \right\}$$

We claim that $I = \bigoplus_{m \in \mathbb{N}} I_m$ is the desired element of U_b. We first show that I is an ideal. Let $r, s \in I_m$. Then clearly,

$$\frac{(r+s)^{2\deg(b)}}{b^{2m}} \in \mathfrak{q}$$

Since $\mathfrak{q}$ is prime, it then follows that

$$\frac{(r+s)^{\deg(b)}}{b^m} \in \mathfrak{q}$$

And so I_m is an abelian group. It then follows immediately that I is a homogeneous ideal. To see that it is a prime ideal, suppose that $rs \in I$ for some homogeneous elements $r, s \in S$. Then

$$\frac{(rs)^{\deg(b)}}{b^{\deg(rs)}} = \frac{r^{\deg(b)} s^{\deg b}}{b^{\deg(r)} b^{\deg(s)}} = \frac{r^{\deg(b)}}{b^{\deg(r)}} \cdot \frac{s^{\deg(b)}}{b^{\deg(s)}}$$

From this we see that either $r \in I$ or $s \in I$ so I is prime. Now clearly, $b \notin I$ so, indeed, $I \in D_+(b)$. It then follows immediately that $f(I) = \mathfrak{q}$ thereby proving that f is bijective.

We now show that f is a homeomorphism. Note that $D_+(b) \cap V_+(I)$ for homogeneous ideals $I \triangleleft S$ are the closed sets of $D_+(b)$. Then

$$f(D_+(b) \cap V_+(I)) = V(I_b \cap S_{(b)})$$

The other direction is also clear so f is a homeomorphism.

We next show that there exists an isomorphism $\varphi : \mathcal{O}_{U_b}(U) \to \mathcal{O}_Y(f(U))$ for all open sets $U \subseteq U_b$. Observe that by Part 1, we have isomorphisms

$$(\mathcal{O}_X)_{\mathfrak{p}} \cong S_{(\mathfrak{p})} \cong (S_{(b)})_{f(\mathfrak{p})} \cong (\mathcal{O}_Y)_{f(\mathfrak{p})}$$

where the middle isomorphism is given by

$$\frac{a}{c} \mapsto \frac{a}{1} \Big/ \frac{c}{1}$$

This then induces an isomorphism on the level of sections and we are done.

Part 3: This follows from Part 1 and Part 2. Note that the condition $S_+ \not\subseteq \mathfrak{p}$ ensures that the open sets $D_+(b)$ cover $X = \mathrm{Proj}(S)$.

□

Example 2.2.12. Let R be a ring and $S = R[t_0, \ldots, t_n]$. Then S is a graded ring with homogeneous components S_d consisting of all homogeneous polynomials of degree d. We define **n-projective space** over R to be

$$\mathbb{P}^n_R = \mathrm{Proj}(S)$$

The open sets $D_+(t_0), \ldots, D_+(t_n)$ cover $\mathbb{P}^n_R$. By the above Theorem, we have that

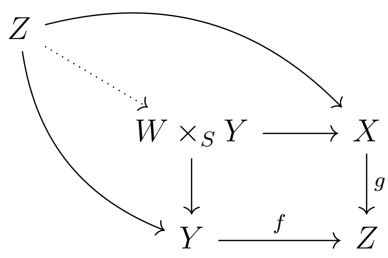
$$D_+(t_i) \cong \mathrm{Spec}(S_{(t_i)}) \cong R\left[\frac{t_0}{t_i}, \ldots, \frac{t_n}{t_i}\right] \cong \mathrm{Spec}(\mathbb{A}^n_R)$$

2.3 Fibred Products

Proposition 2.3.1. *Let X be a topological space. Then* $\mathbf{Sch}(X)$ *has pullbacks (fibred products). In other words, given a commutative diagram*

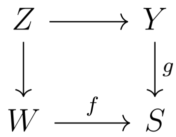
$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ W & \xrightarrow{f} & S \end{array}$$

of schemes over X, there exists a unique scheme, denoted $W \times_S Y$ such that we have a commutative diagram

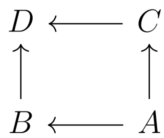


and a unique morphism of schemes $Z \to WX_SY$. Categorically, $W \times_S Y$ is universal amongst all schemes Z that complete the above diagram to a commutative diagram.

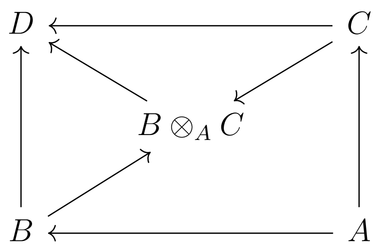
Proof. First suppose that all schemes involved are affine so that $S = \text{Spec}(A)$, $W = \text{Spec}(B)$ and $Y = \text{Spec}(C)$ for some rings A, B and C. Let $Z = \text{Spec}(D)$ for some ring D. A commutative diagram



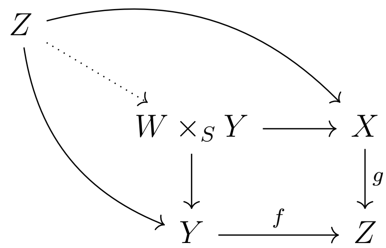
yields a commutative diagram of rings



by reversing the direction of the arrows. By the universal property of tensor products, there exists a unique homomorphism of A-modules $B \otimes_A C \to D$ such that the diagram



commutes. Define $X \times_S Y = \text{Spec}(B \otimes_A C)$. Then we get a commutative diagram



as desired. The proof of the general case is omitted. □

Definition 2.3.2. Let X and Y be schemes and $f : X \to Y$ be morphisms. Given $y \in Y$, let $\mathfrak{m}_y$ be the maximal ideal of $\mathcal{O}_y$ and $k(y) = \mathcal{O}_y/\mathfrak{m}_y$ the residue field of y in Y. We define the **fibre** of f over y to be

$$X_y = \text{Spec}(k(y)) \times_Y X$$

Furthermore, if Y is integral and η is the generic point of Y then we say that X_η is a **generic** fibre of f.

Example 2.3.3. Let $R = \mathbb{C}[t_1, t_2, t_3]/(t_2t_3 - t_1)$ and $X = \text{Spec}(R)$. The homomorphism of rings

$$\begin{aligned} \mathbb{C}[u] &\to R \\ u &\mapsto [t_3] \end{aligned}$$

induces a morphism of schemes $X \to Y = \mathrm{Spec}(\mathbb{C}[u]) = \mathbb{A}^1_{\mathbb{C}}$. Let $y = (u-a) \lhd \mathbb{C}[u]$. We have that

$$k(y) = \mathcal{O}_y/\mathfrak{m}_y \cong \frac{\mathbb{C}[u]_{(u-a)}}{(u-a)_{(u-a)}} \cong \mathbb{C}[u]_{(u-a)} \cong \mathbb{C}$$

The fibre X_y is given by

$$\begin{aligned} X_y &= \mathrm{Spec}\left(\frac{\mathbb{C}[u]}{(u-a)} \otimes_{\mathbb{C}[u]} R\right) \\ &\cong \mathrm{Spec}\left(\frac{R}{(u-a)R}\right) \\ &\cong \frac{\mathbb{C}[t_1,t_2]}{(at_2 - t_1^2)} \end{aligned}$$

In particular, if $a = 0$, $X_y = \mathrm{Spec}\left(\frac{\mathbb{C}[t_1,t_2]}{(t_1^2)}\right)$ which is not reduced.

2.4 $\mathcal{O}_X$-modules

Definition 2.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ a sheaf of modules. We say that $\mathcal{F}$ is an $\boldsymbol{\mathcal{O}_X}$**-module** if for all open sets $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and for all inclusions of open sets $V \subseteq U$ and $s \in \mathcal{O}_X(U), m \in \mathcal{F}(U)$ we have $(sm)|_V = s|_V \cdot m|_V$.

Definition 2.4.2. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. A **morphism** of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that for all open sets $U \subseteq X, \mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Remark.

- If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules then $\ker\varphi$ and $\mathrm{im}\,\varphi$ are $\mathcal{O}_X$-modules.
- If $\mathcal{F}_i$ is a family of $\mathcal{O}_X$-modules then $\bigoplus_i \mathcal{F}_i$ is an $\mathcal{O}_X$-module defined to be the sheafification of the presheaf given by $\bigoplus \mathcal{F}_i(U)$.
- If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is an $\mathcal{O}_X$-module defined to be the sheafification of the presheaf given by $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.
- If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces and $\mathcal{F}$ is an $\mathcal{O}_X$-module then $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module.

Definition 2.4.3. Let $X = \mathrm{Spec}(R)$ be an affine scheme and M an R-module. We define the $\mathcal{O}_X$-module $\widetilde{M}$ by

$$\widetilde{M}(U) = \left\{ s : U \to \dot{\bigcup_{\mathfrak{p} \in U}} M_{\mathfrak{p}} \;\middle|\; \begin{array}{c} \forall\, \mathfrak{p} \in U, s(\mathfrak{p}) \in M_{\mathfrak{p}} \\ \exists \text{ open } \mathfrak{p} \in W \subseteq U \text{ such that } \forall\, \mathfrak{q} \in W, \\ s(\mathfrak{q}) = \dfrac{m}{a} \in M_{\mathfrak{q}} \text{ where } m \in M, a \in R \end{array} \right\}$$

Theorem 2.4.4. *Let $X = \mathrm{Spec}(R)$ be an affine scheme and M an R-module. Then*

1. *$\widetilde{M}$ is indeed an $\mathcal{O}_X$-module.*
2. *$(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ for all $\mathfrak{p} \in X$.*

3. $\widetilde{M}(D(b)) \cong M_b$.

4. $\widetilde{M}(X) \cong M$.

Proof. All proved in the same way as for the case where $M = R$. □

Remark. Let $X = \operatorname{Spec}(R)$. If $M \to N$ is a homomorphism of R-modules then we get a morphism of $\mathcal{O}_X$-modules $\widetilde{M} \to \widetilde{N}$. So if

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a complex of R-modules we then have a complex of sheaves

$$0 \longrightarrow \widetilde{K} \longrightarrow \widetilde{M} \longrightarrow \widetilde{N} \longrightarrow 0$$

Where the first complex is exact if and only if the second complex is exact. Indeed, the complex of R-modules is exact if and only if

$$0 \longrightarrow K_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}} \longrightarrow 0$$

is exact for all $\mathfrak{p} \in X$. This is exact if and only if

$$0 \longrightarrow \widetilde{K}_{\mathfrak{p}} \longrightarrow \widetilde{M}_{\mathfrak{p}} \longrightarrow \widetilde{N}_{\mathfrak{p}} \longrightarrow 0$$

is exact for all $\mathfrak{p} \in X$. This is exact if and only if the original complex of sheaves is exact.

Definition 2.4.5. Let $f : X \to Y$ be a map of topological spaces and $\mathcal{G}$ a sheaf on Y. We define the **inverse image** of $\mathcal{G}$ under f, denoted $f^{-1}\mathcal{G}$, to be the sheafification of the presheaf given by

$$U \mapsto \varinjlim_{V \supseteq F(U)} \mathcal{G}(V)$$

where $U \subseteq X$ is open.

Remark. Elements of the direct limit can be represented by equivalence classes of pairs $[V, t]$ where $f(U) \subseteq V$ and $t \in \mathcal{G}(V)$ and the equivalence relation is given by $(V, t) \sim (V', t)$ if and only if there exists an open $f(U) \subseteq W \subseteq V \cap V'$ such that $t|_W = t'|_W$.

Definition 2.4.6. Let $f : X \to Y$ be a morphism of ringed spaces and $\mathcal{G}$ an $\mathcal{O}_Y$-module. We define the **pullback** of $\mathcal{G}$ under f, denoted $f^*\mathcal{G}$, to be

$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

Theorem 2.4.7. *Let $\alpha : R \to S$ be a ring homomorphism and $f : X = \operatorname{Spec}(S) \to Y = \operatorname{Spec}(R)$ the induced morphism of schemes.*

1. *If M and N are R-modules then*

$$\widetilde{M} \otimes_{\mathcal{O}_Y} \widetilde{N} \cong \widetilde{M \otimes_R N}$$

2. *If $\{ M_i \}$ is a family of R-modules then*

$$\bigoplus \widetilde{M_i} = \widetilde{\bigoplus M_i}$$

3. *If L is an S-module then $f_*\widetilde{L} \cong \widetilde{{}_RL}$ where ${}_RL$ is L considered as an R-module via α.*

4. *If M is an R-module then $f^*\widetilde{M} \cong \widetilde{S \otimes_R M}$.*

Proof. We give the proof of Part 1. Part 2 is analogous and the others are omitted.

Let $\mathcal{F}$ be the presheaf given by $\mathcal{F}(U) = \widetilde{M}(U) \otimes_{\mathcal{O}_Y(U)} \widetilde{N}(U)$. We shall construct an isomorphism of sheaves $\varphi : \mathcal{F} \to \widetilde{M \otimes_R N}$. Fix an open subset $U \subseteq X$ and choose $s \in \widetilde{M}(U)$ and $t \in \widetilde{N}(U)$. Define

$$\begin{aligned} r : U &\to \dot{\bigcup_{\mathfrak{p} \in U}} (M \otimes_R N)_{\mathfrak{p}} = \dot{\bigcup_{\mathfrak{p} \in U}} M_{\mathfrak{p}} \otimes_R N_{\mathfrak{p}} \\ \mathfrak{p} &\mapsto s(\mathfrak{p}) \otimes t(\mathfrak{p}) \end{aligned}$$

If s is locally given by m/a and t is locally given by n/b then r is locally given by $(m \otimes n)/ab$. Now, the mapping $(s, t) \to r$ is bilinear and hence induces a homomorphism of R-modules

$$\varphi_U : \mathcal{F}(U) \to \widetilde{M \otimes_R N}(U)$$

This then induces a morphism of presheaves $\varphi : \mathcal{F} \to \widetilde{M \otimes_R N}$ which in turn gives rise to a morphism of sheaves $\varphi^+ : \mathcal{F}^+ \to \widetilde{M \otimes_R N}$.

Given $\mathfrak{p} \in X$, we have that

$$\varphi^+_{\mathfrak{p}} = \varphi_{\mathfrak{p}} : \mathcal{F}_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \to \widetilde{M \otimes_R N}_{\mathfrak{p}} = (M \otimes_R N)_{\mathfrak{p}}$$

is an isomorphism at the level of stalks. This then implies that φ is an isomorphism and we are done. □

2.5 Quasi-coherent sheaves

Definition 2.5.1. Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is **quasi-coherent** if for all open affine $U = \operatorname{Spec}(R) \subseteq X$, $\mathcal{F}|_U = \widetilde{M}$ for some R-module M. Furthermore, we say that $\mathcal{F}$ is **coherent** if M can be chosen to be finitely generated over R.

Example 2.5.2. Let X be a scheme. Then $\mathcal{O}_X$ is coherent. Indeed, for all open affine sets $U = \operatorname{Spec}(R)$ we have $\mathcal{O}_X|_U = \widetilde{R}$.

Example 2.5.3. Let R be a discrete valuation ring and set $X = \operatorname{Spec}(R) = \{0, \mathfrak{m}\}$. Define an $\mathcal{O}_X$-module $\mathcal{G}$ of X by setting $\mathcal{F}(\{0\}) = \operatorname{Frac}(R)$ and $\mathcal{F}(X) = 0$. Then $\mathcal{G}$ is not quasi-coherent. Indeed, if $U \subseteq X$ is open affine containing $\mathfrak{m}$ then $U = X$. If $\mathcal{G}$ were to be quasi-coherent, we would have that $\mathcal{G} = \widetilde{M}$ for some R-module M. But then $M = \mathcal{F}(X) = 0$ which is a contradiction.

Lemma 2.5.4. *Let $X = \operatorname{Spec}(R)$ be an affine scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Let $M = \mathcal{F}(X)$. Then there exists a natural morphism of $\mathcal{O}_X$-modules $f : \widetilde{M} \to \mathcal{F}$.*

Proof. For all $a \in R$, define a homomorphism

$$\begin{aligned} M_a &\to \mathcal{F}(D(a)) \\ \frac{m}{a^r} &\to \frac{1}{a^r} \cdot m|_{D(a)} \end{aligned}$$

This induces a morphism of $\mathcal{O}_X$-modules $\widetilde{M} \to \mathcal{F}$. Now, each open set $U \subseteq X$ is covered by open sets of the form $D(a_i)$. For each section $s \in \widetilde{M}(U)$, consider images of $s|_{D(a_i)}$ and glue them together to get a section in $\mathcal{F}(U)$ and call it image of s. □

Corollary 2.5.5. *Let $X = \mathrm{Spec}(R)$ be an affine scheme and M an R-module. If $a \in R$ then*

$$\widetilde{M}|_{D(a)} \cong \widetilde{M_a}$$

as $\mathcal{O}_X$-modules.

Proof. By Lemma 2.5.4, we have a morphism of $\mathcal{O}_X$-modules

$$\varphi : \widetilde{M_a} \to \widetilde{M}|_{D(a)}$$

Now, for all $\mathfrak{p} \in D(a)$ we have that $\varphi_{\mathfrak{p}} : (\widetilde{M})_a)_{\mathfrak{p}} \to (\widetilde{M}|_{D(a)})_{\mathfrak{p}}$ is an isomorphism. This implies that φ itself is an isomorphism. □

Definition 2.5.6. Let X be a scheme. We say that X is **Noetherian** if X can be covered by finitely many open affine subschemes $U_1, \ldots, U_r$ such that for all i, $U_i = \mathrm{Spec}(R_i)$ for some Noetherian R_i.

Theorem 2.5.7. *Let X be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. If $U = \mathrm{Spec}(R) \subseteq X$ is open affine then $\mathcal{F}|_U \cong \widetilde{M}$ for some R-module M. Furthermore, if X is Noetherian and $\mathcal{F}$ is coherent, M can be chosen to be finitely generated.*

Proof. Fix an open affine set $U = \mathrm{Spec}(R) \subseteq X$. By definition, for all $x \in U$, there exists an open affine neighbourhood of $X, V = \mathrm{Spec}(B)$ such that $\mathcal{F}|_V \cong \widetilde{N}$ for some B-module N. We can always find a $b \in B$ such that $x \in D_V(b)$ where $D_V(b)$ is understood as taking the open set $D(b)$ with respect to V. By the previous corollary, we have that $\mathcal{F}|_{D(b)} \cong \widetilde{N_b}$ so we may assume that $V \subseteq U$. This allows us to replace X with U and so we can just suppose that $X = \mathrm{Spec}(R)$ is affine.

Write $X = \bigcup D(a_i)$ as a finite union such that $\mathcal{F}|_{D(a_i)} \cong \widetilde{M_i}$ for some R_{a_i}-module M_i. Now, denote $f_i : D(a_i) \hookrightarrow X, f_{ij} : D(a_i a_j) \hookrightarrow X, \mathcal{G} = \bigoplus_i (f_i)_* \mathcal{F}|_{D(a_i)}$ and $\mathcal{H} = \bigoplus_{i,j} (f_{ij})_* \mathcal{F}|_{D(a_i a_j)}$. Consider the sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

where φ_U is the homomorphism given by $s \mapsto (s|_{U \cap D(a_i)})_i$ and ψ_U is the homomorphism given by $(s_i) \mapsto (s_i|_{U \cap D(a_i a_j)} - s_j|_{U \cap D(a_i a_j)})_{i,j}$. Then the exactness of this sequence follows from the fact that $\mathcal{F}$ is a sheaf.

Note that $\mathcal{F}_{D(a_i)} \cong \widetilde{M_i}$ and $\mathcal{F}|_{D(a_i a_j)} \cong \widetilde{M_{i,j}}$ for some $A_{a_i a_j}$-module M_{ij}. Moreover, $(f_i)_* \widetilde{M_i} = {}_R\widetilde{M_i}$ and $(f_{ij})_* \widetilde{M_{ij}} = {}_R\widetilde{M_{ij}}$. The exact sequence is thus

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \bigoplus_i \widetilde{{}_R M_i} \xrightarrow{\psi} \bigoplus_{i,j} \widetilde{{}_R M_{i,j}}$$

Taking global sections of the exact sequence, we thus have a second exact sequence

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\varphi_X} \bigoplus_i {}_R M_i \longrightarrow \bigoplus_{i,j} {}_R M_{i,j}$$

Taking $\sim$, we then get an exact sequence

$$0 \longrightarrow \widetilde{\mathcal{F}(X)} \xrightarrow{\varphi_X} \bigoplus_i \widetilde{{}_R M_i} \longrightarrow \bigoplus_{i,j} \widetilde{{}_R M_{i,j}}$$

Hence $\mathcal{F} \cong \ker \varphi \cong \widetilde{\mathcal{F}}$ and we are done. The statement for coherent $\mathcal{O}_X$-modules on Noetherian schemes follows by the same argumentation. □

Theorem 2.5.8. *Let X be a scheme and $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of quasi-coherent $\mathcal{O}_X$-modules. Then $\ker\varphi$ and $\operatorname{im}\varphi$ are quasi-coherent. Furthermore, if X is Noetherian and $\mathcal{F}$ and $\mathcal{G}$ are coherent then $\ker\varphi$ and $\operatorname{im}\varphi$ are coherent.*

Proof. Let $U = \operatorname{Spec}(R) \subseteq X$ be an open affine set. By Theorem 2.5.7 $\mathcal{F}|_U \cong \widetilde{M}$ and $\mathcal{G}|_U \cong \widetilde{N}$ for some R-modules M and N. Then φ induces a homomorphism of R-modules $\beta : M = \mathcal{F}(U) \to N = \mathcal{G}(U)$. Let $K = \ker\beta$. We have an exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\ \varphi\ } N$$

Passing to $\sim$, we get an exact sequence

$$0 \longrightarrow \widetilde{K} \longrightarrow \widetilde{M} \xrightarrow{\ \varphi|_U\ } \widetilde{N}$$

And so $(\ker\varphi)|_U \cong \widetilde{K}$ and $\ker\varphi$ is quasi-coherent. A similar argument proves the result for $\operatorname{im}\varphi$ and the Noetherian case. □

Theorem 2.5.9. *Let $f : X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{G}$ an $\mathcal{O}_Y$-module. We have that*

1. *if $\mathcal{G}$ is quasi-coherent then $f^*\mathcal{G}$ is quasi-coherent.*
2. *if $\mathcal{G}$ is coherent then $f^*\mathcal{G}$ is coherent.*
3. *if $\mathcal{F}$ is quasi-coherent and*
   - *for all $y \in Y$ there exists an open affine neighbourhood of y $W \subseteq Y$ such that $f^{-1}W = \bigcup_{i=1}^n U_i$ for some open affine U_i.*
   - *for all i, j, $U_i \cap U_j = \bigcup_{k=1}^m U_{i,j,k}$ for some open affine $U_{i,j,k}$.*

   then $f_*\mathcal{F}$ is quasi-coherent.

Proof.

Part 1: Since quasi-coherency is a local property, we may assume that Y is affine. Then $\mathcal{G}$ is given by some R-module M. If $U = \operatorname{Spec}(B) \subseteq X$ is open affine, Theorem 2.4.7 implies that

$$f^*\mathcal{G}|_U \cong \widetilde{M \otimes_R B}$$

which is a B-module and so $f^*\mathcal{G}$ is quasi-coherent.

Part 2: We follow the same argumentation as above. Since $f^*\mathcal{G}$ is coherent, M is finitely generated over R. Hence $M \otimes_R B$ is finitely generated over B and $f^*\mathcal{G}$ is coherent.

Part 3: As usual, we may assume that Y is affine. Let $f_i : U_i \hookrightarrow X, f_{i,j,k} : U_{i,j,k} \hookrightarrow X, \mathcal{G} = \bigoplus_{i=1}^n (f_i)_*(\mathcal{F}|_{U_i})$ and $\mathcal{H} = \bigoplus_{i,j,k} (f_{i,j,k})_*(\mathcal{F}|_{U_{i,j,k}})$. We then have a sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \varphi\ } \mathcal{G} \xrightarrow{\ \psi\ } \mathcal{H}$$

where φ_U is given by $s \mapsto (s|_{U_i})_i$ and ψ_U is given by $(s_i)_i \mapsto (s_i|_{U_{i,j,k}} - s_j|_{U_{i,j,k}})$. Then this sequence is exact since $\mathcal{F}$ is a sheaf. Taking pushforwards yields an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{\ \varphi\ } f_*\mathcal{G} \xrightarrow{\ \psi\ } f_*\mathcal{H}$$

Note that

$$f_*\mathcal{G} = \bigoplus_i (f_*)(f_i)_*(\mathcal{F}|_{U_i})$$

and similarly for $f_*\mathcal{H}$. This implies that both $f_*\mathcal{G}$ and $f_*\mathcal{H}$ are quasi-coherent as they are both given by modules as a result of Theorem 2.4.7. $f_*\mathcal{F}$ is thus the kernel of a morphism of quasi-coherent $\mathcal{O}_X$-modules whence Theorem 2.5.8 implies that $f_*\mathcal{F}$ is quasi-coherent. □

Definition 2.5.10. Let X be a scheme. An **ideal** sheaf I of X is a subsheaf $I \subseteq \mathcal{O}_X$.

Theorem 2.5.11. *Let X be a scheme. Then there is a one-to-one correspondence between the quasi-coherent ideal sheaves of X and the closed subschemes of X. Moreover, if X is Noetherian then the same is true for coherent ideal sheaves.*

Proof. Let Y be a closed subscheme of X and let $f : Y \to X$ be a representative closed immersion of Y. By definition, we have that f maps Y homeomorphically onto a closed subset of X and that the corresponding morphism of sheaves $\varphi : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is a surjection. Let $\mathcal{I} = \ker\varphi$. Then $\mathcal{I}$ is clearly an ideal sheaf. We claim that $\mathcal{I}$ is in fact quasi-coherent. Now, $\mathcal{O}_X$ is itself quasi-coherent so by Theorem 2.5.9, it suffices to show that $f_*\mathcal{O}_Y$ is quasi-coherent.

Assume that $X = \operatorname{Spec}(R)$ is affine. Let $\{ U_i \}$ be an open affine covering of Y and choose open affine $W_i \subseteq X$ such that $U_i = Y \cap W_i$ where Y is identified with a closed subset of X via f. We can cover X and, in particular, each W_i, by open affine sets of the form $D(b)$ so that we have a family of elements $\{ b_\alpha \}$ such that for all α either $D(b_\alpha) \subseteq X\backslash Y$ or $D(b_\alpha) \subseteq W_i$ for some i. Since $X = \bigcup_\alpha D(b_\alpha)$, we have that $\sum(b_\alpha) = R$. Indeed, if this weren't the case then $\sum(b_\alpha)$ would be contained in some maximal ideal of R which is prime and thus not contained in any of the $D(b_\alpha)$. $\sum(b_\alpha)$ is thus finitely generated as an ideal and we may assume that there are only finitely many of the b_α, say $b_1, \ldots, b_n$. Now, for all α, $f^{-1}D(b_\alpha)$ is an open affine subscheme of some U_i and thus of Y. Furthermore, $f^{-1}D(b_\alpha) \cap f^{-1}D(b_\beta) = f^{-1}D(b_\alpha b_\beta)$ and so the conditions of Part 3 of Theorem 2.5.9 are satisfied whence $f_*\mathcal{O}_Y$ is quasi-coherent.

Conversely, let $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal sheaf. For all open affine sets $U = \operatorname{Spec}(R)$, we have that $\mathcal{I}|_U = \widetilde{I}$ for some ideal $I \lhd R$. Indeed, the R-modules contained in R are exactly the ideals of R. We shall construct a corresponding closed subscheme of X locally. Given an open affine set $U \subseteq X$ such that $\mathcal{I}|_U = \widetilde{I}$, define $Y_U = V_U(I) := \{ \mathfrak{p} \in V(I) \mid \mathfrak{p} \in U \}$. Let Y be the union of all such Y_U; this set shall be the topological structure of the closed subscheme. We must first check that Y is well-defined - it is not yet clear that on $U \cap U'$ this construction is independent of working with either U or U'. In other words, given open affine sets $U = \operatorname{Spec}(R), U' = \operatorname{Spec}(R') \subseteq X$, we must check that $Y_U \cap U' = Y_{U'} \cap U$. To this end, choose $\mathfrak{p} \in Y_U \cap U'$. Since $U \cap U'$ is again affine, there exists some $b' \in R'$ such that $\mathfrak{p} \in D_{U'}(b') \subseteq U$. Now, $\mathcal{O}_{U'}(D_{U'}(b')) = R'_{b'}$ and $\mathcal{O}_U(U) = R$ so we get a homomorphism of rings $\theta : R \to R'_{b'}$. On the other hand, we have the canonical homomorphism $R' \to R'_{b'}$. Then $\langle \theta(I) \rangle = I'_{b'}$. Hence if $I \subseteq \mathfrak{p}$ then $I'_{b'} \subseteq \mathfrak{p}$ whence $I \subseteq \mathfrak{p}$ so that $\mathfrak{p} \in Y_{U'} \cap U$. By symmetry, it then follows that $Y_U \cap U' = Y_{U'} \cap U$ for all affine sets $U, U' \subseteq X$.

Let $\mathcal{G}$ denote the sheafification of the presheaf given by $U \mapsto \mathcal{O}_X(U)/\mathcal{I}(U)$. Since $Y \subseteq X$ is a closed subspace, it follows that $\mathcal{G}|_{X\backslash Y} = 0$. Hence $\mathcal{G} = f_*\mathcal{O}_Y$ for some sheaf $\mathcal{O}_Y$ where $f : Y \hookrightarrow X$ is the inclusion.

In particular, $\mathcal{O}_Y$ is given on open sets $W \subseteq Y$ by writing $Y = U \cap X$ for some open set U of X and setting $\mathcal{O}_Y = \mathcal{G}(U)$. This is well-defined since $\mathcal{G}|_{X\backslash Y} = 0$. Moreover, let

$x \in Y \subseteq X$. Choose an affine set $U \subseteq X$ so that $U = \operatorname{Spec} R$ and $\mathcal{I}(U) = I \triangleleft R$. Then $(Y \cap U, \mathcal{O}_{Y \cap U}) = \operatorname{Spec}(R/I)$ so that Y is a scheme. Hence by construction we have an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y \longrightarrow 0$$

which implies that $f : Y \hookrightarrow X$ is a closed immersion and so Y is a closed subscheme. □

2.6 Sheaves Associated to Graded Modules

Definition 2.6.1. Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring and M an S-module. We say that M is **graded** if there exist a family of S-submodules of M $\{ M_d \}_{d \in \mathbb{Z}}$ such that

$$M = \bigoplus_{d \in \mathbb{Z}} M_d$$

and $S_d \cdot M_e \subseteq M_{d+e}$.

Definition 2.6.2. Let $X = \operatorname{Proj}(S)$ be a projective scheme and M a graded S-module. We define the $\mathcal{O}_X$-module $\widetilde{M}$ by

$$\widetilde{M}(U) = \left\{ s : U \to \dot{\bigcup_{\mathfrak{p} \in U}} M_{\mathfrak{p}} \;\middle|\; \begin{array}{c} \forall\, \mathfrak{p} \in U, s(\mathfrak{p}) \in M_{\mathfrak{p}} \\ \exists \text{ open } \mathfrak{p} \in W \subseteq U \text{ such that } \forall\, \mathfrak{q} \in W, \\ s(\mathfrak{q}) = \dfrac{m}{a} \in M_{\mathfrak{q}} \text{ where } m \in M, a \in R \\ \text{are homogeneous of the same degree} \end{array} \right\}$$

Remark. Let $X = \operatorname{Proj}(S)$ be a projective scheme. Then $\mathcal{O}_X \cong \widetilde{S}$.

Theorem 2.6.3. *Let $X = \operatorname{Proj}(S)$ be a projective scheme. Then*

1. $(\widetilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$ *for all* $\mathfrak{p} \in X$.
2. $\widetilde{M}|_{D_+(b)} \cong \widetilde{M_{(b)}}$ *considered as a sheaf on* $\operatorname{Spec}(S_{(b)})$ *for all homogeneous* $b \in S_+$.
3. $\widetilde{M}$ *is quasi-coherent.*

Proof. The proof for Part 1 and Part 2 are the same as for the case of $M = S$. Part 3 is an immediate consequence of Part 2 since the open sets $D_+(b)$ are a basis for X. □

Definition 2.6.4. Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring and $M = \bigoplus_{d \in \mathbb{Z}} M_d$ a graded S-module. Given $n \in \mathbb{Z}$, let $M(n)$ be the graded S-module whose $\deg d$ piece is M_{d+n}. Moreover, if $X = \operatorname{Proj}(S)$ is a projective scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module, we define

$$\mathcal{O}_X(n) = \widetilde{S(n)}$$
$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

Definition 2.6.5. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$ module $\mathcal{L}$ is said to be **invertible** if for all $x \in X$ there exists an open set $x \in U$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$.

Theorem 2.6.6. *$S = \bigoplus_{d \geq 0} S_d$ be a graded ring which is generated over S_0 (as an S_0-algebra) by elements in S_1 and $M = \bigoplus_{d \in \mathbb{Z}} M_d, N = \bigoplus_{d \in \mathbb{Z}} N_d$ a graded S-modules. Then*

1. $\mathcal{O}_X(n)$ *is invertible for all* $n \in \mathbb{Z}$.

2. $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} = \widetilde{M \otimes_S N}$.

3. $\widetilde{M}(n) \cong \widetilde{M(n)}$.

4. $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$ *for all* $m, n \in \mathbb{Z}$.

Proof.

Part 1: Since S is generated over S_0 by S_1, sets of the form $D_+(b)$ with $b \in S_1$ cover X. Hence, given $b \in S_1$, it suffices to show that $\mathcal{O}_{D_+(b)}(n)$ is invertible for all $n \in \mathbb{Z}$.

To this end, fix $b \in S_1$ and $n \in \mathbb{Z}$. We have

$$\mathcal{O}_X|_{D_+(b)} = \widetilde{S(n)}|_{D_+(b)} \cong \widetilde{S(n)_{(b)}}$$

Now, we have an isomorphism

$$\begin{aligned} S(n)_{(b)} &\to S_{(b)} \\ \frac{a}{b^r} &\mapsto \frac{a}{b^{r+n}} \end{aligned}$$

So that

$$\mathcal{O}_X(n)|_{D_+(b)} \cong \widetilde{S_{(b)}} \cong \mathcal{O}_{D_+(b)}$$

Part 2: We construct an isomorphism of $\mathcal{O}_X$-modules

$$\varphi : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_S N}$$

Since S is generated over S_0 as an S_0-algebra by elements of S_1, it suffices to define φ on open sets $D_+(b)$ for $b \in S_1$. Observe that we have

$$\begin{aligned} (\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})(D_+(b)) &\cong (\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})|_{D_+(b)}(D_+(b)) \\ &= (\widetilde{M_{(b)}} \otimes_{D_+(b)} \widetilde{N(b)})(D_+(b)) \\ &\cong M_{(b)} \otimes_{S_{(b)}} N_{(b)} \end{aligned}$$

Moreover, we have

$$\widetilde{M \otimes_S N}(D_+(b)) \cong (M \otimes_S N)_{(b)}$$

Now note that we have a canonical isomorphism

$$\begin{aligned} M_{(b)} \otimes_{S_{(b)}} N_{(b)} &\to (M \otimes_S N)_{(b)} \\ \frac{m}{b^n} \otimes \frac{n}{b^{n'}} &\mapsto \frac{m \otimes n}{b^{n+n'}} \end{aligned}$$

since the tensor product commutes with localisation. We can thus define $\varphi_{D_+(b)}$ to be this isomorphism and we are done.

Part 3: By Part 2 we have

$$\begin{aligned} \widetilde{M}(n) &= \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \\ &= \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{S(n)} \\ &\cong \widetilde{M \otimes_S S(n)} \end{aligned}$$

Now note that we have an isomorphism

$$\begin{aligned} M \otimes_S S(n) &\to M(n) \\ m \otimes a &\mapsto am \end{aligned}$$

so that

$$\widetilde{M}(n) \cong \widetilde{M \otimes_S S(n)} \cong \widetilde{M(n)}$$

Part 4: By Part 2 we have

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{S(m)} \otimes_{\mathcal{O}_X} \widetilde{S(n)} \cong \widetilde{S(m) \otimes_S S(n)}$$

Now note that we have an isomorphism

$$\begin{aligned} S(m) \otimes S(n) &\to S(m+n) \\ a \otimes b &\mapsto ab \end{aligned}$$

so that

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \widetilde{S(m) \otimes_S S(n)} \cong \widetilde{S(m+n)}$$

□

Lemma 2.6.7. *Let $X = \mathrm{Proj}(T)$ and $Y = \mathrm{Proj}(S)$ be projective schemes and $\alpha : S \to T$ a homomorphism of graded rings. Then α induces a morphism of schemes $f : U \to Y$ where*

$$U = \{ \mathfrak{p} \in \mathrm{Proj}(T) \mid \alpha^{-1}(\mathfrak{p}) \in \mathrm{Proj}(S) \}$$

Moreover, if α is surjective then this morphism in fact a closed immersion $f : X \to Y$.

Proof. Let $S = \bigoplus_{d\geq 0} S_d$ and $T = \bigoplus_{d \geq 0} T_d$ and define

$$\begin{aligned} f : U &\to Y \\ \mathfrak{q} &\mapsto \alpha^{-1}(\mathfrak{q}) \end{aligned}$$

which is well-defined since α preserves degrees. To show that this map is continuous, it suffices to show that $f^{-1}(D_+(b))$ is open for all homogeneous $b \in S$. But

$$f^{-1}(D_+(b)) = (\alpha^{-1})^{-1}(D_+(b)) = U \cap D_+(\alpha(b))$$

which is clearly open. We must now define a morphism of sheaves $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_U$. To this end, we must provide a homomorphism of rings $\varphi_V : \mathcal{O}_Y(V) \to (f_*\mathcal{O}_U)(V) = \mathcal{O}_U(f^{-1}V)$ for each open set $V \subseteq Y$. Once again, it suffices to provide a homomorphism of rings

$$\varphi_{D_+(b)} : \mathcal{O}_Y(D_+(b)) \to \mathcal{O}_U(f^{-1}(D_+(b))) = \mathcal{O}_U(U \cap D_+(\alpha(b))) = \mathcal{O}_X(U \cap D_+(\alpha(b)))$$

for each homogeneous $b \in S$. Observe that we have a natural homomorphism of rings

$$\mathcal{O}_Y(D_+(b)) = S_{(b)} \to T_{(\alpha(b))} = \mathcal{O}_X(D_+(\alpha(b)))$$

induced by α. Composing this homomorphism with the restriction to U provides us with the desired homomorpism. To show that it is indeed a morphism of sheaves, we need to show that the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \longrightarrow & \mathcal{O}_Y(W) \\ \downarrow{\scriptstyle \varphi_V} & & \downarrow{\scriptstyle \varphi_V} \\ (f_*\mathcal{O}_U)(V) & \longrightarrow & (f_*\mathcal{O}_U)(W) \end{array}$$

commutes. But this is clear by construction. If α is surjective then $U = X$ and we get a morphism of schemes $f : X \to Y$. Letting $I = \ker \alpha$ we then have an exact sequence

$$0 \longrightarrow I \longrightarrow S \longrightarrow T \cong S/I \longrightarrow 0$$

which yields an exact sequence of sheaves

$$0 \longrightarrow \widetilde{I} \longrightarrow \widetilde{S} \longrightarrow \widetilde{T} \longrightarrow 0$$

with $\widetilde{I}$ an ideal sheaf of $\mathcal{O}_Y = \widetilde{S}$. We thus have a closed immersion $f : X \to Y$ and so X is a closed subscheme of Y. $\square$

Theorem 2.6.8. *Let $S = \bigoplus_{d \geq 0} S_d$ and $T = \bigoplus_{d \geq 0} T_d$ such that S is generated as an S_0-algebra by S_1. Let $X = \operatorname{Proj}(S)$ and $Y = \operatorname{Proj}(T)$ be the corresponding projective schemes and suppose we are given a surjective ring homomorphism $\alpha : S \to T$ with $f : Y \to X$ the corresponding morphism of schemes.*

1. *If L is a graded S-module then $f^*\widetilde{L} \cong \widetilde{L \otimes_S T}$.*
2. *If K is a graded T-module then $f_*\widetilde{K} \cong \widetilde{{}_SK}$ where ${}_SK$ is K considered as a graded S-module via α.*

In particular, we have $f^*\mathcal{O}_X(n) \cong \mathcal{O}_Y(n)$ and $f_*\mathcal{O}_Y(n) \cong (f_*\mathcal{O}_Y)(n) \cong (f_*\mathcal{O}_Y) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Proof. We shall construct a morphism of $\mathcal{O}_X$-modules $\psi : f^*\widetilde{L} \to \widetilde{L \otimes_S T}$. It suffices to construct an isomorphism on open sets of the form $D_+(c) \subseteq Y$ where $c \in T_1$. Let $b \in S_1$ be such that $\alpha(b) = c$. Expanding definitions, we see that

$$\begin{aligned} f^*(\widetilde{L}(D_+(c))) &= f^*(\widetilde{L}|_{D_+(b)})(D_+(c)) \\ &= f^*(\widetilde{L_{(b)}})(D_+(c)) \\ &= \widetilde{L_{(b)} \otimes_{S_{(b)}} T_{(c)}}(D_+(c))) \\ &\cong L_{(b)} \otimes_{S_{(b)}} T_{(c)} \end{aligned}$$

On the other hand, we have

$$\widetilde{L \otimes_S T}(D_+(c)) = (L \otimes_S T)_{(c)}$$

Now, we have an isomorphism

$$\begin{gathered} L_{(b)} \otimes_{S_{(b)}} T_{(c)} \to (L \otimes_S T)_{(c)} \\ \frac{l}{b^r} \otimes \frac{t}{c^{r'}} \mapsto \frac{l \otimes t}{c^{r+r'}} \end{gathered}$$

so we have an isomorphism $\psi_{D_+(c)} : (f^*\widetilde{L})(D_+(c)) \to \widetilde{L \otimes_S T}(D_+(c))$ which induces an isomorphism ψ_V for all open sets $V \subseteq Y$ and so an isomorphism of $\mathcal{O}_X$-modules ψ. A similar argument proves that $f_*\widetilde{K} \cong \widetilde{{}_SK}$. Finally,

$$f^*\mathcal{O}_X(n) \cong f^*\widetilde{S(n)} \cong \widetilde{S(n) \otimes_S T} = \widetilde{T(n)} = \mathcal{O}_Y(n)$$

via the isomorphism

$$\begin{aligned} S(n) \otimes_S T &\to T(n) \\ a \otimes t &\mapsto at \end{aligned}$$

and

$$f_* \mathcal{O}_Y(n) \cong f_* \widetilde{T(n)} \cong \widetilde{{}_S T(n)} \cong \widetilde{{}_S T \otimes_S S(n)} \cong f_* \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

via the isomorphism

$$\begin{aligned} {}_S T \otimes_S S(n) &\to {}_S T(n) \\ t \otimes a &\mapsto at \end{aligned}$$

□

3 Divisors and Differentials

3.1 Invertible Sheaves and Cartier Divisors

Definition 3.1.1. Let $(X, \mathcal{O}_X)$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is **locally free of rank n** if for all $x \in X$ there exists an open $x \in U \subseteq X$ such that

$$\mathcal{F}|_U \cong \bigoplus_{i=1}^{n} \mathcal{O}_U$$

Note that if $n = 1$ then this is just the definition of an invertible $\mathcal{O}_X$-module.

Definition 3.1.2. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules. We define an $\mathcal{O}_X$-module $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ whose sections are given by

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \{\, \varphi : \mathcal{F} \mid_U \to \mathcal{G} \mid_U \mid \varphi \text{ is a morphism of } \mathcal{O}_X\text{-modules} \,\}$$

Proposition 3.1.3. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules. Then $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is indeed an $\mathcal{O}_X$-module.*

Proof. We must first show that $\mathcal{H} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups. Indeed, fix an open set $U \subseteq X$. We define an abelian group structure on $\mathcal{H}(U)$ as follows. Given two morphisms $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ and $\psi : \mathcal{F}|_U \to \mathcal{G}|_U$ we define

$$(\varphi + \psi)|_V = \varphi|_V + \psi|_V$$

for all open sets $V \subseteq U$. This is a well-defined morphism $(\varphi + \psi) : \mathcal{F} \to \mathcal{G}$ since φ and ψ are morphisms of sheaves. The identity morphism $e : \mathcal{F}|_U \to \mathcal{G}|_U$ is given by the trivial morphism $e_V : \mathcal{F}|_U(V) \to \mathcal{G}|_U(V)$. Given a morphism $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$, its inverse $\varphi^{-1} : \mathcal{F}|_U \to \mathcal{G}|_U$ is given pointwise by

$$\begin{aligned} \varphi_V^{-1} : \mathcal{F}|_U(V) &\to \mathcal{G}|_U(V) \\ s &\mapsto \varphi_V(s)^{-1} \end{aligned}$$

Hence $\mathcal{H}(U)$ is indeed an abelian group for all open sets $U \subseteq V$. Now, given open sets $U \subseteq V \subseteq$, we define the restriction morphisms $|_V$ by sending a section $\varphi : \mathcal{F}|_U \to \mathcal{F}|_U$ to $\varphi|_V \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_V, \mathcal{G}|)V)$. $\mathcal{H}$ is thus a presheaf of abelian groups.

We next verify that $\mathcal{H}$ is a sheaf. To this end, fix an open subset $U \subseteq X$ and an open covering $U = \bigcup_i U_i$. Let $\varphi_i \in \mathcal{H}(U_i)$ be sections such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. We need to show that there exists a unique $\varphi \in \mathcal{H}(U)$ such that $\varphi|_{U_i} = \varphi_i$. Observe that, given an open subset $V \subseteq U$, $A_i = V \cap U_i$ cover V. Now fix a section $s \in \mathcal{F}|_U(V)$ and denote $s_i = s|_{A_i}$. For each i we have a morphism

$$\begin{aligned} \varphi_i|_{A_i} : \mathcal{F}|_U(A_i) &\to \mathcal{G}|_U(A_i) \\ s_i &\mapsto t_i \end{aligned}$$

By the compatibility of φ on overlaps, the t_i are also compatible on overlaps. Since $\mathcal{G}|_U$ is a sheaf, there exists a unique $t \in \mathcal{G}|_U(V)$ such that $t|_{A_i} = t_i$ for each i. We can then define

$$\begin{aligned} \varphi_V : \mathcal{F}|_U(V) &\to \mathcal{G}|_U(V) \\ s &\mapsto t \end{aligned}$$

Now, by construction, $\varphi|_{U_i} = \varphi_i$ and so φ is the desired section $\varphi \in \mathcal{H}(U)$. Hence $\mathcal{H}$ is a sheaf of abelian groups.

It remains to show that $\mathcal{H}$ is an $\mathcal{O}_X$-module. To this end we must show that, for all open subsets $U \subseteq X$, $\mathcal{H}(U)$ is an $\mathcal{O}_X(U)$-module. As we have shown, it is an abelian group so we just need to endow it with a $\mathcal{O}_X(U)$-module strucutre. Fix a section $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ and an element $r \in \mathcal{O}_X(U)$. Define $r \cdot \varphi$ to be the morphism that is given pointwise by

$$\begin{aligned} (r \cdot \varphi)_V : \mathcal{F}|_U(V) &\to \mathcal{G}|_U(V) \\ s &\mapsto r|_V \cdot \varphi(s) \end{aligned}$$

To verify that this indeed gives us an $\mathcal{O}_X(U)$-module structure, fix $\phi, \psi \in \mathcal{H}(U)$ and a section $s \in F|_U(V)$. Then

$$\begin{aligned} (r \cdot (\varphi + \psi))|_V(s) = r|_V \cdot (\varphi + \psi)(s) = r|_V \cdot (\varphi(s) + \psi(s)) &= r|_V \cdot \varphi(s) + r|_V \psi(s) \\ &= (r \cdot \varphi)|_V + (r \cdot \psi)|_V \end{aligned}$$

The other module axioms follow similarly. □

Lemma 3.1.4. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also an invertible $\mathcal{O}_X$-module.*

Proof. Fix $x \in X$. We need to exhibit an open neighbourhood $x \in W \subseteq X$ such that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)|_W \cong \mathcal{O}_W$. Since $\mathcal{L}$ is invertible, there exists an open neighbourhood $x \in W \subseteq X$ such that $\mathcal{L}|_W \cong \mathcal{O}_W$. Then

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)|_W = \operatorname{Hom}_{\mathcal{O}_W}(\mathcal{L}|_W, \mathcal{O}_W) \cong \operatorname{Hom}_{\mathcal{O}_W}(\mathcal{O}_W, \mathcal{O}_W) - \mathcal{O}_W$$

so W is a suitable choice of neighbourhood. □

Theorem 3.1.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the set of invertible sheaves (up to isomorphism) on X is an abelian group called the* ***Picard group*** *of X and denoted $\operatorname{Pic}(X)$.*

Proof. We define the group operation on $\operatorname{Pic}(X)$ to be the tensor product of $\mathcal{O}_X$-modules which is clearly a commutative binary operation. We first check that, given $\mathcal{L}, \mathcal{M} \in \operatorname{Pic}(X)$ we have $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \in \operatorname{Pic}(X)$. Indeed for all $x \in X$ there exists an open neighbourhood $x \in U \subseteq X$ such that $\mathcal{L}|_U = \mathcal{O}_U$ and an open neighbourhood $x \in V \subseteq X$ such that $\mathcal{M}|_V = \mathcal{O}_V$. Let $W = U \cap V$. Then

$$(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M})|_W \cong \mathcal{O}_W \otimes_{\mathcal{O}_W} \mathcal{O}_W \cong \mathcal{O}_W$$

The identity element is clearly $\mathcal{O}_X$ since

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{L}$$

Given $\mathcal{L} \in \mathrm{Pic}(X)$, we claim that the inverse of $\mathcal{L}$ is given by $\mathcal{L}^{-1} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. To this end, we shall construct an isomorphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{O}_X$. We define φ pointwise by

$$\begin{aligned} \varphi_U : \mathcal{L}^{-1}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}(U) &\to \mathcal{O}_X(U) \\ \psi \otimes t &\mapsto \psi_U(t) \end{aligned}$$

Since for every $x \in X$ we can find an open neighbourhood $x \in W \subseteq X$ such that $\mathcal{L}|_W \cong \mathcal{O}_W \mathcal{L}^{-1}|_W$, we get an induced isomorphism of stalks so ϕ must be an isomorphism.

Finally, the associativity of the binary operation is immediate from the associativity of tensor products of modules. □

Definition 3.1.6. Let X be an integral scheme, η its unique generic point and $K = \mathcal{O}_\eta$ its function field so that we have an injective ring homomorphism $\mathcal{O}_X(U) \hookrightarrow K$ for all open $U \subseteq X$. We define a **Cartier divisor** to be a system of the form $\{ (U_i, f_i) \}_{i \in I}$ where the U_i give an open covering of X and $f_i \in K$ is such that f_i/f_j and f_j/f_i are both in $\mathcal{O}_X(U_i \cap U_j)$.

We define an equivalence relation $\sim$ on the set of all Cartier divisors by declaring that $(U_i, f_i) \sim (U_\alpha, g_\alpha)$ if and only if for all i, α we have that f_i/g_α is invertible in $\mathcal{O}_X(U_i \cap V_\alpha)$.

A Cartier divisor D is said to be **principal** if it is represented by a single pair (X, f) for some $f \in K$. In this case, we write $D \sim 0$. Given two Cartier divisors E and F represented by (U_i, f_i) and (V_α, g_α) respectively, we define $E + F$ to be the divisor given by the system $U_i \cap V_\alpha, f_i g_\alpha)$ and $-E$ the divisor given by the system $(U_i, 1/f_i)$. If $E - F \sim 0$ then we write $E \sim F$.

We define the **Cartier divisor class group**, denoted $\mathrm{Div}(X)$, to be the free abelian group on the set of Cartier divisors modulo the equivalence relation $\sim$.

Definition 3.1.7. Let X be an integral scheme and K its function field. Given a Cartier divisor $D = (V_i, f_i)$, we define an $\mathcal{O}_X$-module

$$\mathcal{O}_X(D)(U) = \{ h \in K \mid hf_i \in \mathcal{O}_X(U \cap V_i) \}$$

Lemma 3.1.8. *Let X be an integral scheme and K its functon field. Let D be a Cartier divisor for X. Then $\mathcal{O}_X(D)$ is indeed an $\mathcal{O}_X$-module.*

Proof. We must first show that this definition is independent of the choice of representative of D. Indeed, let $D = (V_i, f_i)$ and $D' = (W_\alpha, g_\alpha)$ be two representatives of D (slightly abusing notation). We want to show that $\mathcal{O}_X(D) = \mathcal{O}_X(D')$. Fix an open set $U \subseteq X$ and $h \in \mathcal{O}_X(D)(U)$. By definition, h is an element of K such that $hf_i \in \mathcal{O}_X(U \cap V_i)$ for all i. Since D and D' define the same divisor, we have that f_i/g_α is invertible in $\mathcal{O}_X(U_i \cap V_\alpha)$ for all i, α. Then

$$\begin{aligned} hf_i \in \mathcal{O}_X(U \cap V_i) &\implies hf_i \cdot \frac{g_\alpha}{f_i} \in \mathcal{O}_X(U \cap V_i \cap W_\alpha) \text{ for all } i, \alpha \\ &\implies hg_\alpha \in \mathcal{O}_X(U \cap W_\alpha) \text{ for all } \alpha \\ &\implies h \in \mathcal{O}_X(D')(U) \end{aligned}$$

Hence $\mathcal{O}_X(D) \subseteq \mathcal{O}_X(D')$. By symmetry it then follows that $\mathcal{O}_X(D) = \mathcal{O}_X(D')$.

It is clear that $\mathcal{O}_X(D)(U)$ is an abelian group under addition and that it inherits the restriction morphisms from $\mathcal{O}_X$ and is thus a presheaf. To see that it is a sheaf, let $U = \bigcup_i U_i$ be an open cover and $h_i \in \mathcal{O}_X(D)(U_i)$ such that $h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j}$. We need to show that there exists a unique $h \in \mathcal{O}_X(D)(U)$ such that $h|_{U_i} = h_i$. Fixing m, observe that $\{ U_i \cap V_m \}_{i \in I}$ is an open cover of $U \cap V_m$. Then $h_i f_m$ are compatible on overlaps since the h_i are. Since $\mathcal{O}_X$ is a sheaf, there exists a unique $h' \in U \cap V_m$ such that $h'|_{U_i} = h_i f_m$. Defining $h = h' f_m^{-1} \in K$ shows that $h f_m \in \mathcal{O}_X(U \cap V_m)$. Indeed, if this were not the case then we would have that $(hf_m)|_{U_i} = h_i f_m \notin \mathcal{O}_X(U_i \cap V_m)$ which is a contradiction. Now by the definition of a Cartier divisor, we have

$$\begin{aligned} hf_m \in \mathcal{O}_X(U \cap V_m) &\implies hf_m \cdot \frac{f_{m'}}{f_m} \in \mathcal{O}_X(U \cap V_m \cap V_{m'}) \\ &\implies hf_{m'} \in \mathcal{O}_X(U \cap V_{m'}) \end{aligned}$$

so that $h \in \mathcal{O}_X(D)(U)$. Finally, $\mathcal{O}_X(D)$ clearly inherits an $\mathcal{O}_X$-module structure as a subset of $K = \mathcal{O}_X(U)$. □

Theorem 3.1.9. *Let X be an integral scheme and K its function field. If D and E are Cartier divisors on X then*

1. *$\mathcal{O}_X(D)$ is invertible.*
2. *$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \cong \mathcal{O}_X(D+E)$.*
3. *$\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^{-1}$.*
4. *$D \sim E$ if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$.*

Proof.

Part 1: Suppose that D is represented by (U_i, f_i). We have isomorphisms

$$\mathcal{O}_X(D)|_{U_i} \cong \mathcal{O}_{U_i} \cdot \frac{1}{f_i} \cong \mathcal{O}_{U_i}$$

Part 2: Define an isomorphism

$$\begin{aligned} \psi_U : \mathcal{O}_X(D)(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(E)(U) &\to \mathcal{O}_X(D+E)(U) \\ h \otimes h' &\mapsto hh' \end{aligned}$$

on open sets $U \subseteq X$. To see that this is well-defined, suppose that (U_i, f_i) represents D and (V_α, g_α) represents E. Since $h \in \mathcal{O}_X(D)$ we have $hf_i \in \mathcal{O}_X(U \cap U_i)$ for all i. Similarly, $h' \in \mathcal{O}_X(E)$ so that $h' g_\alpha \in \mathcal{O}_X(U \cap V_\alpha)$ for all α. Then $hh' f_i g_\alpha \in \mathcal{O}_X(U \cap U_i \cap V_\alpha)$ for all i and α. Hence $hh' \in \mathcal{O}_X(D+E)(U)$.

Now, all $\mathcal{O}_X$-modules are invertible so we can find a common open set U such that

$$\mathcal{O}_X(D)|_U \cong \mathcal{O}_X(E)|_U \cong \mathcal{O}_X(D_E)|_U \cong \mathcal{O}_X|_U$$

Hence we have an induced isomorphism of stalks for every $x \in X$ whence they must be isomorphic.

Part 3: By the previous Part, we have

$$\mathcal{O}_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \cong \mathcal{O}_X(-D+D) \cong \mathcal{O}_X(0) \cong \mathcal{O}_X$$

But inverses are unique in Pic(X) so we must have that $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^{-1}$.

Part 4: It suffices to show that $D \sim 0$ if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$. To this end, first suppose that $D \sim 0$ so that D is represented by (X, f). Then $\mathcal{O}_X(D) \cong \mathcal{O}_X \cdot \frac{1}{f} \cong \mathcal{O}_X$.

Conversely, suppose that we have an isomorphism $\varphi : \mathcal{O}_X \to \mathcal{O}_X(D)$ and that D is represented by (U_i, f_i). Let $f \in \mathcal{O}_X(D)(X)$ be the image of $1 \in \mathcal{O}_X(X)$ under φ_X. Then $\mathcal{O}_X(D)|_U = \mathcal{O}_U \cdot \frac{1}{f}$.

On the other hand, $\mathcal{O}_X(D)|_{U_i} = \mathcal{O}_{U_i} \cdot \frac{1}{f_i}$. Hence f/f_i is invertible in $\mathcal{O}_X(U_i)$ for all i so that D is represented by (X, f) whence $D \sim 0$.

□

Remark. This Theorem provides an injection $\mathrm{Div}(X) \to \mathrm{Pic}(X)$.

3.2 Differential Forms

Definition 3.2.1. Let R be a ring and S an R-algebra. For all $s \in S$ let ds be a symbol and X the free S-module generated by the ds. Let L be the S-submodule generated by the relations

1. $dr, r \in R$
2. $d(s+t) - ds - dt, s, t \in S$
3. $d(st) - tds - sdt, s, t \in S$

We define the **module of relative differential forms** of S over R to be $\Omega_{S/R} = X/L$.

Remark. Let M be an S-module and $\alpha : S \to M$ a homomorphism such that

- $\alpha(r) = 0$ for all $r \in R$
- $\alpha(s+t) = \alpha(s) + \alpha(t)$
- $\alpha(st) = t\alpha(s) + s\alpha(t)$

Then α necessarily factors uniquely through $\Omega_{S/R}$.

Example 3.2.2. Let $S = R[t_1, \dots, t_n]$ for some commutative ring R. Then $dt_1, \dots, dt_n$ generate $\Omega_{S/R}$ where $d(t_1t_2) = t_2dt_1 + t_1dt_2$. In fact, $dt_1, \dots, dt_n$ generate $\Omega_{S/R}$ freely. Indeed, define a homomorphism

$$\alpha : S \to M = \bigoplus_{i=1}^{n} S \cdot dt_i$$
$$f \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} dt_i$$

then $\alpha(t_i) = dt_i$. The universal property of $\Omega_{S/R}$ then implies that α factors uniquely through $\Omega_{S/R}$, say via $\beta : \Omega_{S/R} \to M$. β is necessarily surjective and M is free so it is infact an isomorphism.

Definition 3.2.3. Let $f : X \to Y$ be a morphism of affine schemes where $X = \mathrm{Spec}(S)$ and $Y = \mathrm{Spec}(R)$. Let $\alpha : R \to S$ be the homomorphism of rings that induces f and consider S as an R-algebra via α. We define the **sheaf of relative differential forms** of Y over X to be $\widetilde{\Omega_{S/R}}$.

If X and Y are arbitrary schemes then we may take an affine open cover $Y = \bigcup_i V_i$ and cover $f^{-1}V_i$ with affine schemes as $f^{-1}V_i = \bigcup_j U_{i,j}$. We then define Ω_{U_{ij}/V_i} as above and glue them together to define a global sheaf $\Omega_{X/Y}$.

Example 3.2.4. Let R be a ring, $S = R[t_1, \ldots, t_n]$, $X = \mathbb{A}^n_R = \mathrm{Spec}(S)$ and $Y = \mathrm{Spec}(R)$. Let $f : X \to Y$ be the morphism of schemes induced by the ring homomorphism

$$\begin{aligned} \alpha : R &\to R[t_1, \ldots, t_n] \\ r &\mapsto r \end{aligned}$$

and consider S as an R-module via α. Then $\Omega_{X/Y} = \widetilde{\Omega_{S/R}} = \widetilde{\bigoplus_{i=1}^n S} = \bigoplus_{i=1}^n \mathcal{O}_X$

Example 3.2.5. Let R be a ring, $S = R[t_0, \ldots, t_n]$, $X = \mathbb{P}^n_R = \mathrm{Proj}(S)$ and $Y = \mathrm{Spec}(R)$. Let $f : X \to Y$ be the morphism of schemes induced by the ring homomorphism

$$\begin{aligned} \alpha : R &\to R[t_0, \ldots, t_n] \\ r &\mapsto r \end{aligned}$$

and consider S as an R-module via α. We can cover X by open affine sets of the form $D_+(t_0), \ldots, D_+(t_n)$ where $D_+(t_i) \cong \mathbb{A}^n_R$. We can glue all the sheaves $\Omega_{D_+(t_i)/Y}$ together to get a sheaf $\Omega_{X/Y}$ such that $\Omega_{X/Y} \cong \bigoplus_{i=1}^n \mathcal{O}_{D_+}(t_i)$.

Theorem 3.2.6. *Let R be a ring, $X = \mathbb{P}^n_R$ and $Y = \mathrm{Spec}(R)$. Then we have an exact sequence*

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \bigoplus_{i=1}^{n+1} \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Proof. Proof omitted (see handwritten Part III notes). □

Example 3.2.7. With assumptions as before, we have that $\Omega(X/Y) = 0$. Indeed, the Theorem gives us an injection

$$\Omega_{X/Y}(X) \hookrightarrow \bigoplus_{i=1}^{n+1} \mathcal{O}_X(-1)(X)$$

But by a question on an example sheet, we know the latter sheaf has no non-trivial global sections.

Example 3.2.8. Let $f : X \to Y$ be a closed immersion. Then $\Omega_{X/Y} = 0$. Indeed, we may assume that X and Y are affine schemes so that $X = \mathrm{Spec}(S), Y = \mathrm{Spec}(R)$ and let f correspond to some ring homomorphism $\alpha : R \to S$ so that $S \cong R/\ker\alpha$. Since α is surjective, it follows that $\Omega_{S/R} = 0$.

4 Cohomology

4.1 Results from Category Theory

Definition 4.1.1. By an **abelian category** we shall mean one of the following

1. **AbGrp** - Category of abelian groups and homomorphisms of groups.
2. $\mathbf{Mod_R}$ - Category of modules over a commutative ring R and R-module homomorphisms.
3. $\mathbf{Sh}(X)$ - Category of sheaves of rings over a topological space X and morphisms of sheaves.
4. $\mathfrak{Mod}(X)$ - Category of $\mathcal{O}_X$-modules over a ringed space $(X, \mathcal{O}_\mathcal{X})$ and morphisms of $\mathcal{O}_X$-modules.
5. $\mathfrak{Qco}(X)$ - Category of quasi-coherent sheaves on a scheme X and morphisms of quasi-coherent sheaves.

Definition 4.1.2. Let $\mathcal{A}$ be an abelian category. By a **complex** we mean a sequence

$$\ldots \longrightarrow A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \longrightarrow \ldots$$

of objects and morphisms in $\mathcal{A}$ such that $\operatorname{im} d^{i-1} \subseteq \ker d^i$. We denote such a sequence by $A^\bullet$.

We define the ***i^th^*-cohomology** object of $A^\bullet$ to be

$$h^i(A^\bullet) = \frac{\ker d^i}{\operatorname{im} d^{i-1}}$$

We say that $A^\bullet$ is **exact** if $h^i(A^\bullet) = 0$ for all i.

Definition 4.1.3. Let $\mathcal{A}$ be an abelian category and $A^\bullet$ and $B^\bullet$ complexes in $\mathcal{A}$. We define a **morphism** of complexes to be morphisms $f_i : A^i \to B^i$ for each i such that the diagrams

$$\begin{array}{ccc} A^i & \xrightarrow{a^i} & A^{i+1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_i} \\ B^i & \xrightarrow{b^i} & B^{i+1} \end{array}$$

commute for all i. Given a sequence

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

of complexes and morphisms between them, we say that such a sequence is **exact** if the sequence

$$0 \longrightarrow A^i \longrightarrow B^i \longrightarrow C^i \longrightarrow 0$$

is exact for every i.

Proposition 4.1.4. *Let $\mathcal{A}$ be an abelian category and*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

an exact sequence of complexes. Then we have a long exact sequence of cohomology groups

$$\begin{array}{ccccccc} 0 \longrightarrow & h^0(A^\bullet) & \longrightarrow & h^0(B^\bullet) & \longrightarrow & h^0(C^\bullet) & \\ \longrightarrow & h^1(A^\bullet) & \longrightarrow & h^1(B^\bullet) & \longrightarrow & h^1(C^\bullet) & \\ \longrightarrow & h^2(A^\bullet) & \longrightarrow & h^2(B^\bullet) & \longrightarrow & h^2(C^\bullet) & \dashrightarrow \\ & \vdots & & & & & \\ \dashrightarrow & h^n(A^\bullet) & \longrightarrow & h^n(B^\bullet) & \longrightarrow & h^n(C^\bullet) & \end{array}$$

Definition 4.1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. We say that a functor $F : \mathcal{A} \to \mathcal{B}$ is additive if for all $A, A' \in \operatorname{ob} \mathcal{A}$ the map $\operatorname{Hom}(A, A') \to \operatorname{Hom}(FA, FA')$ is a homomorphism of abelian groups.

We say that F is **left-exact** if it is additive and for each exact sequence

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$$

we have an exact sequence

$$0 \longrightarrow FA \longrightarrow FA' \longrightarrow FA''$$

Similarly, we have right-exact functors. We say that a functor is **exact** if it is both left and right exact.

Example 4.1.6. We have a left-exact functor

$$F : \mathbf{Sh}(X) \to \mathbf{AbGrp}$$
$$\mathcal{F} \mapsto \mathcal{F}(X)$$

Definition 4.1.7. Let $\mathcal{A}$ be an abelian category. We say that an object $I \in \operatorname{ob} \mathcal{A}$ is **injective** if for every diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & A' \\ & & \downarrow & \swarrow & \\ & & I & & \end{array}$$

with first row exact there exists a morphism $A' \to I$ extending the diagram to a commutative diagram.

Example 4.1.8. $\mathbb{Q}$ is injective in **Grp**.

Definition 4.1.9. Let $\mathcal{A}$ be an abelian category and $A \in \operatorname{ob} \mathcal{A}$ an object. We define a **injective resolution** of A to be a sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

where each I^i is injective. We say that $\mathcal{A}$ **has enough injectives** if every object admits an injective resolution.

Example 4.1.10. Let R be a commutative ring. Then $\mathbf{Mod_R}$ has enough injectives.

Definition 4.1.11. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact covariant functor of abelian categories. We define the **right-derived functors** $R^iF : \mathcal{A} \to \mathcal{B}$ in the following way. For all objects $A \in \operatorname{ob}\mathcal{A}$ choose an injective resolution $I(A)$. Then we define $R^iF(A) = h^i(FI(A))$.

Theorem 4.1.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact covariant functor of abelian categories. Then*

1. *R^iF is independent of the choice of the injective resolution*[2].
2. *$R^0F = F$*
3. *Every exact sequence*

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$$

induces a long exact sequence

$$\begin{array}{l} 0 \longrightarrow R^0F(A) \longrightarrow R^0F(A') \longrightarrow R^0F(A'') \\ \longrightarrow R^1F(A) \longrightarrow R^1F(A') \longrightarrow R^1F(A'') \\ \longrightarrow R^2F(A) \longrightarrow R^2F(A') \longrightarrow R^2F(A'') \\ \quad \vdots \\ \longrightarrow R^nF(A) \longrightarrow R^nF(A') \longrightarrow R^nF(A'') \end{array}$$

4. *For every commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

we have a commutative diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & R^iF(A) & \longrightarrow & R^iF(A') & \longrightarrow & {}^iF(A'') & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & R^iF(B) & \longrightarrow & R^iF(B') & \longrightarrow & R^iF(B'') & \longrightarrow & \dots \end{array}$$

Definition 4.1.13. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact covariant functor of abelian categories. An object $J \in \operatorname{ob}\mathcal{A}$ is said to be **acyclic** if $R^iF(J) = 0$ for all $i > 0$.

Theorem 4.1.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact covariant functor of abelian categories. If*

[2] Injective resolutions are unique up to homotopy and cohomology objects are homotopy-invariant.

$$0 \longrightarrow AJ^0 \longrightarrow J^1 \longrightarrow \dots$$

is an exact sequence with J^i acyclic for all i then

$$R^iF(A) = h^i(0 \to F(J^0) \to F(J^1) \to \dots)$$

Proof. Proof omitted. □

Example 4.1.15. The following are all left-exact functors

1. $\mathbf{Sh}(\mathbf{X}) \to \mathbf{AbGrp} : \mathcal{F} \mapsto \mathcal{F}(X)$
2. $\mathfrak{Mod}(X) \to \mathbf{AbGrp} : \mathcal{F} \mapsto \mathcal{F}(X)$
3. $\mathbf{Mod_R} \to \mathbf{Mod_R} : M \mapsto \mathrm{Hom}_R(L, M)$ for some commutative ring R and R-module L.
4. $\mathbf{Sh}(X) \to \mathbf{Sh}(Y) : \mathcal{F} \mapsto f_*\mathcal{F}$ for some continuous function $f : X \to Y$

4.2 Cohomology of Sheaves

Proposition 4.2.1. *Let X be a topological space. Then $\mathbf{Sh}(X)$ has products and the functor $F : \mathbf{Sh}(X) \to \mathbf{AbGrp}$ reflects them.*

Proof. This is immediate from the definitions. □

Proposition 4.2.2. *Let X be a topological space, $\mathcal{G}$ a sheaf on X and $\{ \mathcal{F}_i \}_{i \in I}$ a family of sheaves on X. Then*

$$\mathrm{Hom}\left(G, \prod_{i \in I} \mathcal{F}_i\right) \cong \prod_{i \in I} \mathrm{Hom}\left(\mathcal{G}, \mathcal{F}_i\right)$$

Proof. Let $\pi_j : \prod_{i \in I} \mathcal{F}_i \to \mathcal{F}_j$ be the j^{th} projection map that the product comes equipped with. Fix an open set $U \subseteq X$ and define

$$\varphi_U : \mathrm{Hom}\left(\mathcal{G}, \prod_{i \in I} \mathcal{F}_i\right)(U) \to \left(\prod_{i \in I} \mathrm{Hom}(G, \mathcal{F}_i)\right)(U)$$
$$\psi \mapsto (\pi_i|_U \circ \psi)_{i \in I}$$

One easily verifies that this is indeed an isomorphism of abelian groups and is compatible with restriction maps. □

Theorem 4.2.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then $\mathfrak{Mod}(X)$ has enough injectives.*

Proof. Fix an $\mathcal{O}_X$-module $\mathcal{F} \in \mathfrak{Mod}(X)$ and $x \in X$. Then $\mathcal{F}_x$ is an $\mathcal{O}_x$ module. Since $\mathbf{Mod}_{\mathcal{O}_\mathbf{x}}$ has enough injectives, we can find an injective $\mathcal{O}_x$-module and an injective homomorphism $\mathcal{F}_x \hookrightarrow I_x$. Let f_x denote the embedding of topological spaces $\{ x \} \hookrightarrow X$. Then I_x can be viewed as a sheaf of modules on the singleton space $\{ x \}$. Define $\mathcal{I} = \prod_{x \in X} f_{x*} I_x$. We claim that $\mathcal{I}$ is injective. First note that, for all sheaves $\mathcal{G} \in \mathrm{ob}\, \mathfrak{Mod}(X)$, Proposition 4.2.2 implies that

$$\mathrm{Hom}(\mathcal{G}, \mathcal{I}) = \prod_{x \in X} \mathrm{Hom}(\mathcal{G}, f_{x*} I_x)$$

On the other hand, it is easy to see that we have an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_{x*}I_x)(X) \cong \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{G}_x, I_x)$$

given by sending a morphism of $\mathcal{O}_x$-modules to the corresponding homomorphism of stalks at x. Now consider a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{G} & \xrightarrow{\varphi} & \mathcal{H} \\ & & \downarrow & & \\ & & \mathcal{I} & & \end{array}$$

Descending to stalks, we have a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{G}_x & \xrightarrow{\varphi_x} & \mathcal{H}_x \\ & & \downarrow & \swarrow & \\ & & \mathcal{I}_x = I_x & & \end{array}$$

But I_x is injective so there must exist a morphism completing the above diagram to a commutative diagram. By the aforementioned isomorphism of Hom-sets, we can lift this homomorphism of $\mathcal{O}_x$-modules to a morphism of $\mathcal{O}_X$-modules to complete the first diagram into a commuatative diagram. Hence $\mathcal{I}$ is injective as claimed.

Now fix an object $\mathcal{F} \in \mathrm{ob}\,\mathfrak{Mod}(X)$. We want to construct an injective resolution for $\mathcal{F}$. By the previous discussion, we can choose an injective object $\mathcal{I}_0$ so that we get a sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0$$

Now set $\mathcal{F}^1 = \mathcal{I}^0/\mathcal{F}$ which is naturally an $\mathcal{O}_X$-module. This gives us a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{F}^1 \longrightarrow 0$$

We may choose an injective object $\mathcal{I}^1$ together with an injective morphism $\mathcal{F}^1 \to \mathcal{I}^1$ so that we get a sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1$$

Continuing in this way, we can construct an injective resolution of $\mathcal{F}$. Hence $\mathfrak{Mod}(X)$ has enough injectives. □

Corollary 4.2.4. *Let X be a topological space. Then* $\mathbf{Sh}(X)$ *has enough injectives.*

Proof. Let $\mathcal{O}_X$ be the constant sheaf on X associated to $\mathbb{Z}$. Then $(X, \mathcal{O}_X)$ is a ringed space and any $\mathcal{F} \in \mathrm{ob}\,\mathbf{Sh}(X)$ is naturally an $\mathcal{O}_X$-module. Applying the Theorem then allows us to construct injective resolutions of sheaves of rings on X. □

Definition 4.2.5. Let X be a topological space and $\mathcal{F} \in \mathbf{Sh}(X)$ a sheaf. Let $F : \mathbf{Sh}(X) \to$ **AbGrp** be the functor sending a sheaf to its corresponding group of global sections. We define the ***i***th**-sheaf cohomology group** to be

$$H^i(X, \mathcal{F}) = R^iF(\mathcal{F})$$

Example 4.2.6. Let $\{x\} = X$ be a singleton space and $F : \mathbf{Sh}(X) \to \mathrm{AbGrp}$ the functor which sends a sheaf to its associated global sections. We claim that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. Indeed, fix a sheaf $\mathcal{F} \in \mathrm{ob}\,\mathbf{Sh}(X)$. Choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

Taking stalks we get an exact sequence

$$0 \longrightarrow \mathcal{F}_x \longrightarrow I_x^0 \longrightarrow I_x^1 \longrightarrow \dots$$

But for a singleton space, stalks coincide with global sections so we infact have an exact sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow I^0(X) \longrightarrow I^1(X) \longrightarrow \dots$$

so that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Example 4.2.7. Let K be a field and $S = K[t_0, t_1]$. Let $X = \mathbb{P}_K^1 = \text{Proj}(S)$. Let $x \in X$ be the point corresponding to the ideal $I = \langle t_1 \rangle$. We have an exact sequence

$$0 \longrightarrow I \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

which yields an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \widetilde{I} \longrightarrow \widetilde{S} \longrightarrow \widetilde{S/I} \longrightarrow 0$$

Letting $f : \{x\} \hookrightarrow X$ be the natural embedding and $\mathcal{I} = \widetilde{I}$ the ideal sheaf corresponding to $\{x\}$, this exact sequence is infact

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow f_* \mathcal{O}_{\{x\}} \longrightarrow 0$$

Note that we have an isomorphism

$$\begin{aligned} S(-1) &\cong I \\ a &\mapsto at_1 \end{aligned}$$

so that we have an isomorphism $\mathcal{I} \cong \mathcal{O}_X(-1)$. The exact sequence then becomes

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow f_* \mathcal{O}_{\{x\}} \longrightarrow 0$$

Passing to cohomology groups yields a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{O}_X(-1)) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, f_* \mathcal{O}_{\{x\}}) \\ \longrightarrow H^1(X, \mathcal{O}_X(-1)) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, f_* \mathcal{O}_{\{x\}}) \end{aligned}$$

Since $\mathcal{O}_X(-1)$ has no global sections, we have that $H^0(\mathcal{O}_X(-1)) = 0$. Moreover, we have $H^0(X, \mathcal{O}_X) = H^0(X, f_* \mathcal{O}_{\{x\}}) = K$.

4.3 Flasque Sheaves

Definition 4.3.1. Let X be a topological space and $\mathcal{F} \in \operatorname{ob}\mathbf{Sh}(X)$. We say that $\mathcal{F}$ is **flasque** if for all open $U \subseteq X$, the restriction morphism $\mathcal{F}(X) \to \mathcal{F}(U)$ is a surjective homomorphism.

Theorem 4.3.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{I} \in \operatorname{ob}\mathfrak{Mod}(X)$ is injective then $\mathcal{I}$ is flasque.*

Proof. Fix an open set $U \subseteq X$ and let $t \in \mathcal{I}(U)$. We need to exhibit an element of $\mathcal{I}(X)$ that maps to t under the restriction morphism $\mathcal{I}(X) \to \mathcal{I}(U)$. Define a sheaf $\mathcal{L}_U$ by

$$\mathcal{L}_U(W) = \begin{cases} 0 & \text{if } W \not\subseteq U \\ \mathcal{O}_X(W) & \text{if } W \subseteq U \end{cases}$$

Clearly, $\mathcal{L}_U$ is a subsheaf of $\mathcal{O}_X$. Now define a morphism of sheaves $\mathcal{L}_U \to \mathcal{I}$ by

$$\varphi_W : \mathcal{L}_U(W) \to \mathcal{I}(W) = \begin{cases} 0 & \text{if } W \not\subseteq U \\ a \mapsto at|_W & \text{if } W \subseteq U \end{cases}$$

We then have a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{L}_U & \longrightarrow & \mathcal{O}_X \\ & & \big\downarrow \varphi_W & \swarrow & \\ & & \mathcal{I} & & \end{array}$$

with first row exact. Since $\mathcal{I}$ is injective, there exists a morphism $\psi : \mathcal{O}_X \to \mathcal{I}$ completing the diagram to a commutative diagram. Since ψ is a morphism of sheaves, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(X) & \xrightarrow{|_U} & \mathcal{O}_X(U) \\ \big\downarrow \psi_X & & \big\downarrow \psi_U \\ \mathcal{I}(X) & \xrightarrow{|_U} & \mathcal{I}(U) \end{array}$$

Chasing $1 \in \mathcal{O}_X(X)$ around the diagram shows that there must exist $s \in \mathcal{I}(X)$ mapping to $t \in \mathcal{I}(U)$ under $|_U$ so that $\mathcal{I}$ is flasque. □

Theorem 4.3.3. *Let X be a topological space and $\mathcal{F} \in \operatorname{ob}\mathbf{Sh}(X)$ a flasque sheaf. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Since $\mathcal{S}\langle(X)$ has enough injectives, we can find an injective sheaf $\mathcal{I}$ and an inclusion morphism $\mathcal{F} \subseteq \mathcal{I}$. Setting $\mathcal{G} = \mathcal{I}/\mathcal{F}$ yields a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

We first claim that $\mathcal{G}$ is flasque. In order to do this, we shall show that we have an exact sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{I}(X) \xrightarrow{\alpha} \mathcal{G}(X) \longrightarrow 0$$

Since taking global sections is left-exact, it suffices to show that α is surjective. Fix $t \in \mathcal{G}(X)$. Since $\varphi : \mathcal{I} \to \mathcal{G}$ is surjective, the corresponding homomorphism of stalks is also surjective. This implies that there exists an open neighbourhood $U \subseteq X$ and en element $s \in \mathcal{I}(U)$ such that $\alpha(s) = t|_U$. Consider pairs (U_1, s_1) and (U_2, s_2) such that $s_i \in \mathcal{I}(U_i)$ and $\alpha(s_i) = t|_{U_i}$. Then $s_1|_{U_1 \cap U_2} - s_2|_{U_1 \cap U_2}$ map to 0 under α. Since the sequence

$$0 \longrightarrow \mathcal{F}(U_1 \cap U_2) \longrightarrow \mathcal{I}(U_1 \cap U_2) \longrightarrow \mathcal{G}(U_1 \cap U_2)$$

is exact, $s_1|_{U_1 \cap U_2} - s_2|_{U_1 \cap U_2} \in \mathcal{F}(U_1)$. Now, $\mathcal{F}$ is flasque so there exists $r \in \mathcal{F}(U_1 \cup U_2)$ such that $r|_{U_1 \cap U_2} = s_1|_{U_1 \cap U_2} - s_2|_{U_1 \cap U_2}$. Then $s_2 + r|_{U_2}$ and s_1 are compatible on overlaps. Indeed

$$(s_2 + r|_{U_2})|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2} + r|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2} + s_1|_{U_1 \cap U_2} - s_2|_{U_1 \cap U_2} = s_1|_{U_1 \cap U_2}$$

Since $\mathcal{I}$ is a sheaf, they glue to give a section $s \in \mathcal{I}(U_1 \cup U_2)$. By construction,

$$\begin{aligned} s|_{U_1} &= s_1 \mapsto t|_{U_1} \\ s|_{U_2} &= s_2 + r|_{U_2} \mapsto t|_{U_2} \end{aligned}$$

and so $s \mapsto t|_{U_1 \cup U_2}$ under α. Now let

$$\mathcal{A} = \{ (U, s) \mid U \subseteq X \text{ open }, s \in \mathcal{I}(U), s \mapsto t|_U \}$$

Define a partial order $\leq$ on $\mathcal{A}$ by declaring $(U, s) \leq (U', s')$ if and only if $U \subseteq U'$ and $s'|_U = s$. By Zorn's Lemma, there exists a maximal element in $\mathcal{A}$, say (U, s). We claim that, in fact, $U = X$. Suppose, for a contradiction, that $U \neq X$. Choose $x \in X \backslash U$ and an open neighbourhood $x \in V \subseteq X$ and $l \in \mathcal{I}(V)$ mapping to $t|_V$ under α. By the previous argumentation, we can construct $m \in \mathcal{I}(U \cup V)$ such that $m|_U = s, m|_V = l$ and $m \mapsto t|_{U \cap V}$. But this contradicts the maximality of (U, s) so we must have that $U = X$ and so $s \in \mathcal{I}(X)$ is the desired element mapping to t under α. Thus α is surjective. Now consider the diagram

$$\begin{array}{ccc} \mathcal{I}(X) & \xrightarrow{\alpha} & \mathcal{G}(X) \\ \downarrow |_W & & \downarrow |_W \\ \mathcal{I}(W) & \xrightarrow{\beta} & \mathcal{G}(W) \end{array}$$

The exact same argumentation shows that β is surjective. Since $\mathcal{I}$ is flasque, it follows that $|_W : \mathcal{G}(X) \to \mathcal{G}(W)$ is surjective when $\mathcal{G}$ flasque as claimed.

We now have a long exact sequence of cohomology groups

$$\begin{array}{l} 0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{G}) \\ \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{I}) \longrightarrow H^1(X, \mathcal{G}) \end{array}$$

Since $\mathcal{I}$ is injective, it admits the trivial injective resolution

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{I} \longrightarrow 0 \longrightarrow \dots$$

so that $H^i(X, \mathcal{I}) = 0$ for all $i > 0$. Since $\alpha : \mathcal{I}(X) \to \mathcal{G}(X)$ is surjective, it then follows that $H^1(X, \mathcal{F}) = 0$. From this it follows that $H^1(X, \mathcal{G}) \cong H^{i+1}(X, \mathcal{F})$ for all $i > 0$. But $\mathcal{G}$ is flasque so, by the same argumentation for $\mathcal{F}$, we see that $H^1(X, \mathcal{G}) = 0$ so that $H^2(X, \mathcal{F}) = 0$ by induction. Continuing in this way using induction we can show that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. □

Corollary 4.3.4. *Let X be a topological space and $\mathcal{F} \in \operatorname{ob} \mathbf{Sh}(X)$ a flasque sheaf. Suppose that $\mathcal{F}$ admits a flasque resolution*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

Then

$$H^i(X, \mathcal{F}) = h^i(0 \to \mathcal{I}^0(X) \to \mathcal{I}^1(X) \to \dots)$$

Proof. Since each $\mathcal{I}^j$ is flasque, Theorem 4.3.3 implies that $H^i(X, \mathcal{I}^j) = 0$ for all $i > 0$, $j \geq 0$. Hence each $\mathcal{I}^j$ is acyclic and so appealing to Theorem 4.1.14 proves the claim. □

Corollary 4.3.5. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. Consider the functor*

$$\begin{aligned} F : \mathfrak{Mod}(X) &\to \mathbf{AbGrp} \\ G &\mapsto G(X) \end{aligned}$$

Then $H^i(X, \mathcal{F})$ is isomorphic to $RF^i(\mathcal{F})$. In other words, cohomology calculated in $\mathrm{Sh}(X)$ *coincides with that calculated in $\mathfrak{Mod}(X)$.*

Proof. Fix an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

in $\mathfrak{Mod}(X)$. By Theorem 4.3.2 this is infact a flasque resolution. Corollary 4.3.4 then implies the assertion of the Corollary. □

4.4 Cohomology of Affine Schemes

Proposition 4.4.1. *Let R be a Noetherian ring and I an injective R-module. Then $\widetilde{I}$ is flasque.*

Proof. Proof omitted. □

Definition 4.4.2. Let X be a scheme and $b \in \mathcal{O}_X(X)$. Define

$$D(b) = \{ x \in X \mid b^{-1} \in \mathcal{O}_x \}$$

Remark. If X is an affine scheme then this coincides with the previous definition of $D(b)$.

Proposition 4.4.3. *Let X be a Noetherian scheme. Then X is affine if and only if there exists $b_1, \dots, b_n \in \mathcal{O}_X(X)$ such that $D(b_i)$ are affine and $\mathcal{O}_X(X) = \langle b_1, \dots, b_n \rangle$.*

Proof. Proof omitted. □

Definition 4.4.4. Let X be a scheme. We say that $x \in X$ is **closed** if $\{ x \}$ is a closed subset of X.

Proposition 4.4.5. *Let X be a Noetherian scheme and $Z \subseteq X$ a closed subset. Then there exists a closed point $x \in Z$.*

Proof. Choose an open affine subset $U \subseteq X$ such that $U \cap Z \neq \varnothing$. If $Z \not\subseteq U$ then replace Z with $Z \cap (X \backslash U)$. Continuining in this way, we can construct a chain of closed subsets

$$\dots \subsetneq Z_2 \subsetneq Z_1$$

But X is Noetherian so this process must terminate and so we can find a closed subset of Z that is contained in U, overloading notation, we also call it Z. Then $Z = \mathrm{Spec}(R)$ for some ring R. Let $\mathfrak{m}$ be any maximal ideal of R. Then $\{ \mathfrak{m} \}$ is a closed subset of Z. Since Z is closed in X, it then follows that $\mathfrak{m}$ is closed in Z so that $\mathfrak{m}$ is a closed point of X. □

Theorem 4.4.6. *Let X be a Noetherian scheme. Then the following are equivalent:*

1. *X is affine.*

2. $H^i(X, \mathcal{F}) = 0$ *for all* $i > 0$ *and quasi-coherent* $\mathcal{F}$*.*

3. $H^1(X, \mathcal{I}) = 0$ *for all coherent ideal sheafs* $\mathcal{I}$*.*

Proof.

(1) $\implies$ (2): First suppose that X is affine so that $X = \operatorname{Spec}(R)$ for some ring R. Fix a quasi-coherent sheaf $\mathcal{F} \in \operatorname{ob} \mathfrak{Qco}(X)$ so that $\mathcal{F} = \widetilde{M}$ for some R-module M. Fix an injective resolution of M

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

in $\mathbf{Mod_R}$. Then

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{I^0} \longrightarrow \widetilde{I^1} \longrightarrow \dots$$

is an flasque resolution of $\mathcal{F}$ in $\mathfrak{Mod}(X)$ by Proposition 4.4.1. Corollary 4.3.4 then implies that

$$\begin{aligned} H^i(X, \mathcal{F}) &= h^i(0 \to \widetilde{I^0}(X) \to \widetilde{I^1}(X) \to \dots) \\ &= h^i(0 \to I^0 \to I^1 \to \dots) \end{aligned}$$

which is exact. Hence $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

(2) $\implies$ (3): This assertion is trivial considering all coherent ideal sheafs are themselves quasi-coherent sheaves.

(3) $\implies$ (1): Fix a closed point $x \in X$ and an open affine set $x \in U$. Let $Y = X \backslash U$ so that both Y and $Y \cup \{ x \}$ are closed. We first claim that any closed set $Z \subseteq X$ can be endowed with the structure of a closed subscheme of X. Indeed, consider the sheaf

$$\mathcal{I}_Z(W) = \{ a \in \mathcal{O}_X(W) \mid a^{-1} \notin \mathcal{O}_z \text{ for all } z \in W \cap Z \}$$

If $W = \operatorname{Spec}(R)$ is open affine then $\mathcal{I}_Z|_W = \widetilde{I}$ where $I \lhd R$ is the largest ideal of R such that $Z \cap W V(I)$. Hence $\mathcal{I}_Z$ is quasi-coherent (in fact, it is coherent since X is Noetherian) and so Z has a closed subscheme structure.

We can apply this construction to the closed sets Y and $Y \cup \{ x \}$ to get closed subschemes $\mathcal{I}_Y$ and $\mathcal{I}_{Y \cup \{ x \}}$. Since $Y \subseteq Y \cup \{ x \}$, we have an inclusion of sheaves $\mathcal{I}_{Y \cup \{ x \}} \subseteq \mathcal{I}_Y$. Letting $\mathcal{L} = \mathcal{I}_Y / \mathcal{I}_{Y \cup \{ x \}}$ we have an exact sequence

$$0 \longrightarrow \mathcal{I}_{Y \cup \{ x \}} \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{L} \longrightarrow 0$$

Since $\mathcal{L}|_{X \backslash \{ x \}} = 0$, it follows that $\mathcal{L}$ is the skyscraper sheaf associated to $\kappa(x)$, the residue field at x. By assumption, we have $H^1(X, \mathcal{I}_{Y \cup \{ x \}}) = 0$ so taking cohomology of the above exact sequence yields an

$$0 \longrightarrow H^0(X, \mathcal{I}_{Y \cup \{ x \}}) \longrightarrow H^0(X, \mathcal{I}_Y) \xrightarrow{\alpha} H^0(X, \mathcal{L}) \longrightarrow 0$$

Since $H^0(X, \mathcal{L}) = \kappa(x)$ and α is surjective so there exists $b \in H^0(X, \mathcal{I}_Y)$ such that $\alpha(b) = 1 \in \kappa(x)$. But this means that any representative of $\alpha(b)$ is invertible in $\mathcal{O}_x$ and so $x \in D(b)$. By construction, $D(b) \subseteq U$. Hence for every closed point $x \in X$, there is a global section $b \in \mathcal{O}_X(X)$ such that $x \in D(b)$. Hence we can construct a family of global sections b_i such that each $D(b_i)$ is affine and $\bigcup_{i \in I} D(b_i)$ contains all closed points of X. In fact, $X = \bigcup_{i \in I} D(b_i)$. Indeed, if this were not the case then $X \setminus \bigcup_{i \in I} D(b_i)$ would be closed and would thus contain a closed point of X which is a contradiction. Since X is Noetherian, we may assume that there are only finitely many such b_i.

We now claim that $\mathcal{O}_X(X)$ is generated by the b_i. We will then be able to conclude that X is affine by Proposition 4.4.3.

Define a morphism of sheaves

$$\varphi_U : \left(\bigoplus_{i=1}^n \mathcal{O}_X\right)(U) \to \mathcal{O}_X(U)$$
$$(s_1, \ldots, s_n) \mapsto \sum_{i=1}^n b_i|_U s_i$$

Let $\mathcal{F}$ be the kernel of this morphism. Then we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow 0$$

φ is surjective since it is locally surjective. Indeed, for all $x \in X$, φ_x is surjective since there exists some b_i which is invertible in $\mathcal{O}_x$. Now define a filtration of length n, denoted $\mathcal{G}_i$, by

$$0 \subseteq \mathcal{O}_X \oplus 0 \cdots \oplus 0 \subseteq \mathcal{O}_X \oplus \mathcal{O}_X \oplus \cdots \oplus 0 \subseteq \cdots \subseteq \bigoplus_{i=1}^n \mathcal{O}_X$$

Then, clearly, $\mathcal{G}_i/\mathcal{G}_{i-1} \cong \mathcal{O}_X$. Let $\mathcal{F}_n = \mathcal{F}$ and inductively define $\mathcal{F}_{i-1} = \ker(\mathcal{F}_i \to \mathcal{G}_i/\mathcal{G}_{i-1})$. We then have exact sequences

$$0 \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_i/\mathcal{F}_{i-1} \longrightarrow 0$$

Moreover, $\mathcal{F}_i/\mathcal{F}_{i-1} \subseteq \mathcal{G}_i/\mathcal{G}_{i-1} \subseteq \mathcal{O}_X$ so that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a coherent ideal sheaf. By hypothesis, we then have that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$. Then $\ker \mathcal{F}_0 = 0$ whence $H^1(X, \mathcal{F}_0) = 0$. By induction, it then follows that $H^1(X, \mathcal{F}_i) = 0$ for all i and, in particular, $H^1(X, \mathcal{F}) = 0$. We then have a short exact sequence of cohomology groups

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, G_n) \xrightarrow{\varphi} H^0(X, \mathcal{O}_X) \longrightarrow 0$$

Hence φ is surjective on global sections whence there exists $(s_1, \ldots, s_n) \in G_n(X)$ such that $1 = \sum_i b_i s_i$ and so $\mathcal{O}_X(X) = (b_1, \ldots, b_n)$. □

4.5 Čech Cohomology

Definition 4.5.1. Let X be a topological space and $\mathcal{F} \in \mathbf{Sh}(X)$ a sheaf. Let $\mathcal{U} = \{ U_i \}_{i \in I}$ be an open covering of X where I is a well-ordered set. Given $i_0, \ldots, i_p \in I$, let $U_{i_0,\ldots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. We define

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$$

Moreover, we define a map $d^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ given by sending $(s_{i_0,\ldots,i_p})$ to $(t_{i_0,\ldots,i_{p+1}})$ where

$$t_{i_0,\ldots,i_{p+1}} = \sum_{l=0}^{p+1} (-1)^l s_{i_0,\ldots,\widehat{i_l},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}$$

where $\widehat{i_l}$ is understood to mean that the i_l-index is dropped. It can be checked that $d^{p+1}d^p = 0$ so that this forms a cochain complex of abelian groups which we refer to as a **Čech complex**. We define the p^{th} Čech cohomology group $\check{H}^p(\mathcal{U}, \mathcal{F})$ to be the p^{th} cohomology group of the aforementioned complex.

Proposition 4.5.2. *Let X be a topological space and $\mathcal{F} \in \mathbf{Sh}(X)$ a sheaf. Let $\mathcal{U} = \{ U_i \}_{i \in I}$ be an open covering of X. Then*

$$\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X) \cong H^0(X, \mathcal{F})$$

Proof. By definition, $\check{H}^0(\mathcal{U}, \mathcal{F}) = \ker d^0$. Now, $C^0(\mathcal{U}, \mathcal{F}) = \prod_{i \in I} \mathcal{F}(U_i)$ and $C^1(\mathcal{U}, \mathcal{F}) = \prod_{i<j} \mathcal{F}(U_i \cap U_j)$. Then

$$\begin{aligned} d^1 : \prod_{i \in I} \mathcal{F}(U_i) &\to \prod_{i<j} \mathcal{F}(U_i \cap U_j) \\ (s_i) &\mapsto ([s_i - s_j]|_{U_i \cap U_j}) \end{aligned}$$

So that $\ker d^0 = \{ (s_i) \mid s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \}$. But this is exactly the global sections of $\mathcal{F}$ since it is a sheaf. □

Example 4.5.3. Let K be a field and $X = \mathbb{P}^1_K = \operatorname{Proj} K[t_0, t_1]$. Consider the open cover $\mathcal{U} = \{ U_0, U_1 \}$ where $U_0 = D_+(t_0), U_1 = D_+(t_1)$. The Čech complex of $\mathcal{O}_X$ is

$$C^\bullet(\mathcal{U}, \mathcal{O}_X) : 0 \longrightarrow C^0(\mathcal{U}, \mathcal{O}_X) \longrightarrow C^1(\mathcal{U}, \mathcal{O}_X) \longrightarrow C^2(\mathcal{U}, \mathcal{O}_X) \longrightarrow \dots$$

Now, $C^p(\mathcal{U}, \mathcal{O}_X) = 0$ for all $p \geq 2$ since there are only two sets in the open cover. Moreover,

$$C^0(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_0) \oplus \mathcal{O}_X(U_1) = K[t_0, t_1]_{(t_0)} \oplus K[t_0, t_1]_{(t_1)}$$

and

$$C^1(\mathcal{U}, \mathcal{O}_X) = \mathcal{O}_X(U_0 \cap U_1) = \mathcal{O}_X(D_+(t_0 t_1)) = K[t_0, t_1]_{(t_0 t_1)}$$

Writing $u = t_1/t_0$ and $v = t_0/t_1$, we first claim that $K[t_0, t_1]_{(t_0)} \cong K[u]$. Indeed, define a homomorphism

$$\begin{aligned} \varphi : K[t_0, t_1]_{(t_0)} &\to K[u] \\ \left[\frac{\sum_{i+j=n} a_{ij} t_0^i t_1^j}{t_0^n} \right] &\mapsto \sum_{i+j=n} a_{ij} u^j \end{aligned}$$

which is clearly well-defined, surjective and injective. The Čech complex is then just

$$0 \longrightarrow K[u] \oplus K[v] \xrightarrow{d^0} K[u, 1/u] \longrightarrow 0$$

$$(f, g) \longmapsto f(u) - g(1/u)$$

so that

$$\begin{aligned} \ker d^0 &= \{ (f, g) \mid f(u) - g(1/u) = 0 \} \\ &= \{ (f, g) \mid f = g \in K \} \cong K \end{aligned}$$

Since d^0 is surjective, it then follows that $\check{H}^p(\mathcal{U}, \mathcal{O}_X) = 0$.

Example 4.5.4. Let K be a field, $X = \mathbb{P}^1_K = \operatorname{Proj} K[t_0, t_1]$ and $Y = \operatorname{Spec} K$. Consider the open cover $\mathcal{U} = \{ U_0, U_1 \}$ where $U_0 = D_+(t_0), U_1 = D_+(t_1)$. The Čech complex of $\Omega_{X/Y}$ is

$$C^\bullet(\mathcal{U},\Omega_{X/Y}) : 0 \longrightarrow C^0(\mathcal{U},\Omega_{X/Y}) \longrightarrow C^1(\mathcal{U},\Omega_{X/Y}) \longrightarrow C^2(\mathcal{U},\Omega_{X/Y}) \longrightarrow \dots$$

Now, $C^p(\mathcal{U},\Omega_{X/Y}) = 0$ for all $p \geq 2$ since there are only two sets in the open cover. Moreover, writing $u = t_1/t_0$ and $v = t_0/t_1$, we have

$$C^0(\mathcal{U},\Omega_{X/Y}) = \Omega_{X/Y}(U_0) \oplus \Omega_{X/Y}(U_1) = K[u]du \oplus K[v]dv$$

and

$$C^1(\mathcal{U},\mathcal{O}_X) = \mathcal{O}_X(U_0 \cap U_1) = K[u,1/u]du$$

so that d^0 is the map

$$(fdu, gdv) \mapsto f(u)du + \frac{1}{u^2}g(1/u)du$$

so that $\ker d^0 = 0$ whence $\check{H}^p(\mathcal{U},\Omega_{X/Y}) = 0$. Moreover, $\operatorname{im} d^0$ contains $u^r \cdot du$ for all $r \in \mathbb{Z}$ except $r = -1$ so that $1/udu \notin \operatorname{im} d^0$. THen

$$\check{H}^1(\mathcal{U},\Omega_{X/Y}) = \frac{\ker d^1}{\operatorname{im} d^0} = \frac{K[u,1/u]du}{\operatorname{im} d^0} \cong K\frac{1}{u}du \cong K$$

Furthermore, $\check{H}^p(\mathcal{U},\Omega_{X/Y}) = 0$ for all $p > 1$.

Example 4.5.5. Let K be a field, $X = \mathbb{P}^1_K = \operatorname{Proj} K[t_0,t_1]$ and $\mathcal{F}$ the constant sheaf associated to $\mathbb{Z}$. Consider the open cover $\mathcal{U} = \{U_0, U_1\}$ where $U_0 = D_+(t_0), U_1 = D_+(t_1)$. The Čech complex of $\mathcal{F}$ is

$$C^\bullet(\mathcal{U},\mathcal{F}) : 0 \longrightarrow C^0(\mathcal{U},\mathcal{F}) \longrightarrow C^1(\mathcal{U},\mathcal{F}) \longrightarrow C^2(\mathcal{U},\mathcal{F}) \longrightarrow \dots$$

Now, $C^p(\mathcal{U},\mathcal{F}) = 0$ for all $p \geq 2$ since there are only two sets in the open cover. Moreover,

$$C^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(U_0) \oplus \mathcal{F}(U_1) = \mathbb{Z} \oplus \mathbb{Z}$$

and

$$C^1(\mathcal{U},\mathcal{F}) = \mathcal{F}(U_0 \cap U_1) = \mathbb{Z}$$

so that d^0 is the map

$$(m,n) \mapsto m - n$$

Now, $\ker d^0 = \{(m,n) \mid m = n\} = \mathbb{Z}$ whence $\check{H}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(X) = \mathbb{Z}$. Moreover, d^0 is surjective so that $\check{H}^p(\mathcal{U},\mathcal{F})$ for all $p > 0$.

Example 4.5.6. Let $X = S^1$ be endowed with the subspace topology from $\mathbb{R}$. Let $\alpha = (0,1)$ and $\beta = (1,0)$ so that $\mathcal{U} = \{U, V\}$ where $U = X \setminus \{\alpha\}$ and $V = X \setminus \{\beta\}$ form an open cover of X. Let $\mathcal{F}$ be the constant sheaf on X associated to $\mathbb{Z}$. The Čech complex of $\mathcal{F}$ is

$$C^\bullet(\mathcal{U},\mathcal{F}) : 0 \longrightarrow C^0(\mathcal{U},\mathcal{F}) \longrightarrow C^1(\mathcal{U},\mathcal{F}) \longrightarrow C^2(\mathcal{U},\mathcal{F}) \longrightarrow \dots$$

Now, $C^p(\mathcal{U},\mathcal{F}) = 0$ for all $p \geq 2$ since there are only two sets in the open cover. Moreover,

$$C^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(U_0) \oplus \mathcal{F}(U_1) = \mathbb{Z} \oplus \mathbb{Z}$$

and

$$C^1(\mathcal{U},\mathcal{F}) = \mathcal{F}(U_0 \cap U_1) = \mathbb{Z} \oplus \mathbb{Z}$$

so that d^0 is the map

$$(m,n) \mapsto (m - n, m - n)$$

We then see that $\ker d^0 \cong \mathbb{Z}$ and $\operatorname{im} d^0 \cong \mathbb{Z}$. So $\check{H}^0(X,\mathcal{F}) = \mathbb{Z}$ and also $\check{H}^1(\mathcal{U},\mathcal{F}) = \mathbb{Z}$. Finally, $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ for all $p > 1$.

4.6 Cohomology of Schemes

Definition 4.6.1. Let X be a topological space, $\mathcal{F} \in \mathbf{Sh}(X)$ a sheaf and $\mathcal{U} = \{ U_i \}_{i \in I}$ an open cover of X for some well-ordered set I. Let $U_{i_0,\dots,i_p} = U_{i_0} \cap \dots \cap U_{i_p}$ and let $f_{i_0,\dots,i_p}$ denote the inclusion map $U_{i_0,\dots,i_p} \hookrightarrow X$. Let $\mathcal{F}_{i_0,\dots,i_p}$ denote the sheaf $(f_{i_0,\dots,i_p})_*(\mathcal{F}|_{U_{i_0,\dots,i_p}})$. Define

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}_{i_0,\dots,i_p}$$

and a map

$$d^p : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$$

pointwise on open $U \subseteq X$ by sending $(s_{i_0,\dots,i_p})$ to $(t_{i_0,\dots,i_{p+1}})$ where

$$t_{i_0,\dots,i_{p+1}} = \sum_{l=0}^{p+1} (-1)^l s_{i_0,\dots,\widehat{i_l},\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}} \cap U}$$

We can similarly check that $d^{p+1} d^p = 0$ so that we get a complex

$$\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) : 0 \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} \dots$$

We extend this to a complex

$$\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) : 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} \dots$$

$$s \in \mathcal{F}(W) \longmapsto (s|_{W \cap U_i})$$

called the **sheaf Čech complex**.

Lemma 4.6.2. *Let X be a topological space, $\mathcal{F} \in \mathbf{Sh}(X)$ a sheaf and $\mathcal{U} = \{ U_i \}_{i \in I}$ an open cover of X for some well-ordered set I. The the sheaf Čech complex of $\mathcal{F}$ is exact.*

Proof. We first claim that

$$0 \longrightarrow \mathcal{F} \xrightarrow{d^{-1}} \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U}, \mathcal{F})$$

is exact by the definition of a sheaf. Indeed, fix an open $W \subseteq X$ and suppose that $(s|_{W \cap U_i}) = 0$. Since $W \cap U_i$ is an open cover of W, the zero sections glue together uniquely to give the zero section in $\mathcal{F}(W)$ so d^{-1} must be injective. To show exactness at $\mathcal{C}^0(\mathcal{U}, \mathcal{F})$, we need to show that $\ker d^0 \subseteq \operatorname{im} d^{-1}$. To this end, fix an open $W \subseteq X$. Suppose that $(s_i) \in \ker d^0$. Then by definition of the differential, we have that

$$(s_i - s_j)|_{U_{i,j} \cap W} = 0$$

But then $s_i|_{U_i \cap U_j \cap W} = s_j|_{U_i \cap U_j \cap W}$ so that the s_i are compatible on overlaps of the open cover $U_i \cap W$ of W. The sheaf axiom then implies that the s_i glue together to give a unique $s \in \mathcal{F}_W$ such that $s|_{U_i \cap W} = s_i$. But then $(s_i) \in \operatorname{im} d^{-1}$ by the definition of d^{-1}.

We now want to show that

$$\mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{p-1}} \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{d^p} \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$$

for all $p \geq 1$. It suffices to show this on the level of stalks. In other words, for all $x \in X$, we need to show that

$$\mathcal{C}^{p-1}(\mathcal{U},\mathcal{F})_x \xrightarrow{d_x^{p-1}} \mathcal{C}^p(\mathcal{U},\mathcal{F})_x \xrightarrow{d_x^p} \mathcal{C}^{p+1}(\mathcal{U},\mathcal{F})_x$$

is exact. Since we are working with stalks, we can throw away any U_i for which $x \notin U_i$ and assume that $X = U_0 = \cdots = U_n$ by replacing X and each U_i with $\bigcap_{i=1}^n U_i$. Now define a map

$$\begin{aligned} e^p : \mathcal{C}^p(\mathcal{U},\mathcal{F})_x &\to \mathcal{C}^{p-1}(\mathcal{U},\mathcal{F})_x \\ [W,(s_{i_0,\ldots,i_p})] &\mapsto [W,(t_{i_0,\ldots,i_{p-1}})] \end{aligned}$$

where

$$t_{i_0,\ldots,i_{p-1}} = \begin{cases} s_{j,i_0,\ldots,i_{p-1}} & \text{if } i_0 \neq j, j = \min I \\ 0 & \text{if } i_0 = j \end{cases}$$

Now, let $\delta_{i_0,j} = 0$ if $i_0 = j$ and 1 otherwise, then

$$\begin{aligned} (d_x^{p-1}e^p + e^{p+1}d_x^p)([W,s_{i_0,\ldots,i_p}]) &= d_x^{p-1}e^p([W,s_{i_0,\ldots,i_p}]) + e^{p+1}d_x^p \\ &= d_x^{p-1}(\delta_{i_0,j}[W,s_{0,i_0,\ldots,i_{p-1}})] + e^{p+1}\sum_{l=0}^{p+2}(-1)^l[W,s_{i_0,\ldots,\widehat{i_l},i_{p+1}}] \\ &= \delta_{i_0,j}\sum_{m=0}^{p}(-1)^m[W,s_{0,i_0,\ldots,\widehat{i_m},\ldots,i_p}] + \delta_{i_0,j}\sum_{l=0}^{p+2}(-1)^l[W,s_{0,i_0,\ldots,\widehat{i_l},\ldots,i_{p+1}}] \\ &= [W,s_{i_0,\ldots,i_p}] \end{aligned}$$

so that $d_x^{p-1}e^p + e^{p+1}d_x^p = \mathrm{id}$. Now fix $[W,s_{i_0,\ldots,i_p}] \in \ker d_x^p$. Applying this formula, we have

$$d_x^{p-1}e^p([W,s_{i_0,\ldots,i_p}]) = [W,s_{i_0,\ldots,i_p}]$$

so that $[W,s_{i_0,\ldots,i_p}] \in \operatorname{im} d_x^{p-1}$. □

Theorem 4.6.3. *Let X be a topological space and $\mathcal{U} = \{U_i\}$ a finite open cover of $\mathcal{X}$. If $\mathcal{F} \in \mathrm{Sh}(X)$ is flasque then*

$$\check{H}^p(\mathcal{U},\mathcal{F}) = 0$$

for all $p > 0$.

Proof. Consider the Čech complex resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U},\mathcal{F}) \longrightarrow \ldots$$

Since $\mathcal{F}$ is flasque, so is $\mathcal{F}|_{U_{i_0,\ldots,i_p}}$ and, in particuar, $\mathcal{F}_{i_0,\ldots,i_p}$ is also flasque. Hence $\mathcal{C}^p(\mathcal{U},\mathcal{F})$ is flasque for all $p \geq 0$ whence the above is a flasque resolution of $\mathcal{F}$. By Corollary 4.3.4 we know that $H^p(X,\mathcal{F})$ is calculated on the sequence

$$0 \longrightarrow \mathcal{C}^0(\mathcal{U},\mathcal{F})(X) \longrightarrow \mathcal{C}^1(\mathcal{U},\mathcal{F})(X) \longrightarrow \ldots$$

On the other hand, the cohomology of the first sequence is $\check{H}^p(\mathcal{U},\mathcal{F})$ by definition and so $\check{H}^p(\mathcal{U},\mathcal{F}) = H^p(\mathcal{U},\mathcal{F})$ by definition. But the latter is 0 by Theorem 4.3.3. □

Theorem 4.6.4. *Let X be a Noetherian scheme such that the intersection of any two open affine subschemes is again affine. Let $\mathcal{U} = \{ U_i \}$ be a finite open affine cover of X. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$$

for all quasi-coherent sheaves $\mathcal{F}$ on X.

Proof. Consider the Čech resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \dots$$

We first claim that $H^l(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = 0$ for all $p \geq 0$ and $l > 0$. It is in fact enough to show that $H^l(X, \mathcal{F}_{i_0,\dots,i_p}) = 0$ for all $p \geq 0$ and $l > 0$. By hypothesis, $U_{i_0,\dots,i_p}$ is affine so Theorem 4.4.6 implies that

$$H^l(U_{i_0,\dots,i_p}, \mathcal{F}|_{U_{i_0,\dots,i_p}}) = 0$$

for all $p > 0$ and $l \geq 0$. By Proposition 4.4.1, we can choose a flasque resolution

$$0 \longrightarrow \mathcal{F}|_{U_{i_0,\dots,i_p}} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

where each $\mathcal{I}^j$ is quasi-coherent. Then $(f_{i_0,\dots,i_p})_*\mathcal{I}^j$ are flasque and quasi-coherent. Then

$$0 \longrightarrow \mathcal{F}_{i_0,\dots,i_p} \longrightarrow (f_{i_0,\dots,i_p}))_*\mathcal{I}^0 \longrightarrow (f_{i_0,\dots,i_p}))_*\mathcal{I}^1 \longrightarrow \dots$$

is also a flasque resolution of $\mathcal{F}_{i_0,\dots,i_p}$. Hence, $H^l(X, \mathcal{F}_{i_0,\dots,i_p}$ are calculated by th complex

$$0 \longrightarrow (f_{i_0,\dots,i_p}))_*\mathcal{I}^0(X) \longrightarrow (f_{i_0,\dots,i_p}))_*\mathcal{I}^1(X) \longrightarrow \dots$$

But this is the same as the complex

$$0 \longrightarrow \mathcal{I}^0(U_{i_0,\dots,i_p}) \longrightarrow \mathcal{I}^1(U_{i_0,\dots,i_p}) \longrightarrow \dots$$

which calculates the cohomology of $H^l(U_{i_0,\dots,i_p}, \mathcal{F}|_{U_{i_0,\dots,i_p}}$. But this is 0 by Theorem 4.4.6. So $H^l(X, \mathcal{F}_{i_0,\dots,i_p}) = 0$ as claimed. This shows that the $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are acyclic with respect to the global section functor so by Theorem 4.1.14 we can calculate $H^l(X, \mathcal{F})$ using the Čech complex of $\mathcal{F}$. This is given by the cohomology of

$$0 \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F})(X) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F})(X) \longrightarrow \dots$$

which is just the ordinary Čech complex

$$0 \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \dots$$

But we know that the cohomology of this is $H^p(X, \mathcal{F}) = \check{H}^p(X, \mathcal{F})$ so we are done. $\square$

Remark. We give a remark on when the conditions of the previous Theorem hold. Let $f : X \to Y$ be a morphism of schemes with $Y = \operatorname{Spec}(R)$ affine. We say that f is **projective** if there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathbb{P}^n_R \\ & \searrow^{f} & \downarrow \\ & & Y \end{array}$$

where g is a closed immersion. If Y is not affine then we can define $\mathbb{P}^n$ over open affine subsets and glue them together. We say that f is **quasi-projective** if there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ & \searrow^{f} & \downarrow \\ & & Y \end{array}$$

with g an open immersion. Now assume that R is Noetherian. Then the intersection of any two open affine subschemes in X is again affine.

5 Cohomology of Projective Schemes

Theorem 5.0.1. *Let K be a field and $X = \mathbb{P}^n_K = \operatorname{Proj}(S)$ where $S = K[t_0, \ldots, t_n]$. Then*

1. *$H^0(X, \mathcal{O}_X(d))$ is the K-vector space generated by all monomials in $t_0, \ldots, t_n$ of degree d.*
2. *$\dim_K H^n(X, \mathcal{O}_X(d)) = \dim_K H^0(X, \mathcal{O}_X(-n-1-d))$.*
3. *$H^p(X, \mathcal{O}_X(d)) = 0$ for all $p > n$.*
4. *$H^p(X, \mathcal{O}_X(d)) = 0$ for all $0 < p < n$.*

Proof.

Part 1: We have that

$$H^0(X, \mathcal{O}_X(d)) = \mathcal{O}_X(d)(X) \cong \{\, (s_i) \mid s_i \in \mathcal{O}_X(d)(U_i), s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \,\}$$

where $U_i = D_+(t_i)$. Now, $\mathcal{O}_X(d)(U_i) = S(d)_{(t_i)}$ so that $s_i \in \mathcal{O}_X(d)(U_i)$ satisfies $s_i = \frac{f_i}{t_i^{e_i}}$ where f_i is homogeneous of degree $d + e_i$ in S. Then

$$\begin{aligned} s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} &\iff \frac{f_i}{t_i^{e_i}} = \frac{f_j}{t_j^{e_j}} \in S(D)_{(t_1 t_2)} \\ &\iff \frac{f_i}{t_i^{e_i}} = \frac{f_j}{t_j^{e_j}} \in S_{(t_1 t_2)} \\ &\iff f_i t_j^{e_j} = f_j t_i^{e_i} \in S \end{aligned}$$

in S. Now, S is a unique factorisation domain so that $t_j^{e_j} \mid f_j$ and $t_i^{e_i} \mid f_i$. Hence there exists a homogeneous $g \in S$ of degree d such that $g = \frac{f_i}{t_i^{e_i}} = s_i$ for all i.

Conversely, given any homogeneous $g \in S$ of degree d, we have a section (s_i) in $\mathcal{O}_X(d)(X)$ given by setting $s_i = \frac{g}{1}$.

Part 2: We shall only prove the case where $-d-n-1 \leq 0$. Now, the group $H^n(X, \mathcal{O}_X(d))$ is calculated by the Čech complex

$$\ldots \longrightarrow C^{n-1}(\mathcal{U}, \mathcal{O}_X(d)) \xrightarrow{d^{n-1}} C^n(\mathcal{U}, \mathcal{O}_X(d)) \xrightarrow{d^n} 0$$

which is just

$$\ldots \longrightarrow \textstyle\prod_{i_0 < \cdots < i_{n-1}} S(d)_{(t_{i_0} \cdots t_{i_{n-1}})} \xrightarrow{d^{n-1}} S(d)_{(t_1 \ldots t_n)} \xrightarrow{d^n} 0$$

where $\mathcal{U} = \{ D_+(t_i) \}_i$. We need to calculate $\operatorname{im} d^{n-1}$. To this end, fix $\sigma \in S(d)_{t_0 \dots t_n}$. We may assume that

$$\sigma = \frac{t_0^{m_0} \dots t_n^{m_n}}{(t_0 \dots t_n)^l}$$

where $\sum_{i=1}^n m_i = d + (n+1)l$. We want to determine when such a σ is not in $\operatorname{im} d^{n-1}$. If there is an i for which $m_i \geq l$ then we would be able to cancel such a t_i from the denominator so that σ would be in the image of the factor of $C^{n-1}(\mathcal{U}, \mathcal{F})$ corresponding to a missing U_i. Moreover, we can assume that $m_i = 0$ for some i, otherwise we may decrease l. Then

$$d + (n+1)l = \sum_{i=0}^{n-1} m_i \leq n(l-1) = nl - n$$

so that $d + l \leq -n$ and so $l \leq -n - d$. But by assumption we have $-n - d \leq 1$ so that $l = 1$. Since each $m_i < l$, the only possibility is then $\sigma = \frac{1}{t_0 \dots t_n}$ which corresponds to the case where $d = -n - 1$. But $\sigma \notin \operatorname{im} d^{n-1}$ so we have

$$\begin{aligned} H^n(X, \mathcal{O}_X(d)) &= \begin{cases} 0 & \text{if } -d-n-1 < 0 \\ K \cdot \frac{1}{t_0 \dots t_n} & \text{if } -d-n-1 = 0 \end{cases} \\ &\cong H^0(X, \mathcal{O}_X(-d-n-1)) \end{aligned}$$

Part 3: This follows immediately from the fact that $H^p(X, \mathcal{O}_X(d)) = \check{H}^p(X, \mathcal{O}_X(d))$. But $C^p(X, \mathcal{O}_X(d)) = 0$ for all $p > n$.

Part 4: We may assume that $n \geq 2$ or there is nothing to prove. Let Y be the closed subscheme defined by $\langle t_n \rangle$ and $g : Y \to \mathbb{P}^n_K$ the corresponding closed immersion. Then $Y \cong \mathbb{P}^{n-1}_K = \operatorname{Proj} K[t_0, \dots, t_{n-1}]$ and we have an exact sequence

$$0 \longrightarrow \widetilde{\langle t_n \rangle} \longrightarrow \mathcal{O}_X \longrightarrow g_* \mathcal{O}_Y \longrightarrow 0$$

Now, we have an isomorphism

$$\begin{aligned} S(-1) &\to \langle t_n \rangle \\ s &\mapsto t_n s \end{aligned}$$

so that $\widetilde{\langle t_n \rangle} = \mathcal{O}_X(-1)$. Hence the exact sequence takes the form

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow g_* \mathcal{O}_Y \longrightarrow 0$$

Tensoring with $\mathcal{O}_X(d)$ yields

$$0 \longrightarrow \mathcal{O}_X(d-1) \longrightarrow \mathcal{O}_X(d) \longrightarrow g_* \mathcal{O}_Y(d) \longrightarrow 0$$

Taking cohomology yields a long exact sequence

$$\begin{aligned}
0 \longrightarrow H^0(X, \mathcal{O}_X(d-1)) &\longrightarrow H^0(X, \mathcal{O}_X(d)) \longrightarrow H^0(X, g_*\mathcal{O}_Y(d)) \\
\longrightarrow H^1(X, \mathcal{O}_X(d-1)) &\xrightarrow{\alpha} H^1(X, \mathcal{O}_X(d)) \longrightarrow H^1(X, g_*\mathcal{O}_Y(d)) \dashrightarrow \\
\dashrightarrow H^{n-1}(X, \mathcal{O}_X(d-1)) &\longrightarrow H^{n-1}(X, \mathcal{O}_X(d)) \longrightarrow H^{n-1}(X, g_*\mathcal{O}_Y(d)) \\
\xrightarrow{\beta} H^n(X, \mathcal{O}_X(d-1)) &\longrightarrow H^n(X, \mathcal{O}_X(d)) \longrightarrow H^n(X, g_*\mathcal{O}_Y(d))
\end{aligned}$$

Now, it is easy to see that $H^p(X, g_*\mathcal{O}_Y(d)) = H^p(Y, \mathcal{O}_Y(d))$ for all $p \geq 0$ since pushing forward a sheaf is an exact functor. Moreover, $H^n(Y, \mathcal{O}_Y(d)) = 0$ by Part 3 so the long exact sequence becomes

$$
\begin{array}{l}
0 \longrightarrow H^0(X, \mathcal{O}_X(d-1)) \xrightarrow{f_1} H^0(X, \mathcal{O}_X(d)) \xrightarrow{f_2} H^0(Y, \mathcal{O}_Y(d)) \\
\xrightarrow{\delta} H^1(X, \mathcal{O}_X(d-1)) \xrightarrow{\alpha} H^1(X, \mathcal{O}_X(d)) \longrightarrow H^1(Y, \mathcal{O}_Y(d)) \dashrightarrow \cdots \\
\dashrightarrow H^{n-1}(X, \mathcal{O}_X(d-1)) \longrightarrow H^{n-1}(X, \mathcal{O}_X(d)) \longrightarrow H^{n-1}(Y, \mathcal{O}_Y(d)) \\
\xrightarrow{\beta} H^n(X, \mathcal{O}_X(d-1)) \xrightarrow{\gamma} H^n(X, \mathcal{O}_X(d)) \longrightarrow 0
\end{array}
$$

Now,

$$
\begin{aligned}
\dim_K(\operatorname{im}\gamma) &= \dim_K H^n(X, \mathcal{O}_X(d)) = \dim_K H^0(X, \mathcal{O}_X(-n-1-d)) = \binom{-2-d}{n-1} \\
\dim_K H^n(X, \mathcal{O}_X(d-1) &= \dim_K H^0(X, \mathcal{O}_X(-n-d) = \binom{-d-1}{n-1}
\end{aligned}
$$

By the Rank-Nullity Theorem, we then have that

$$
\dim_K(\operatorname{im}\beta) = \dim_K(\ker\gamma) = \binom{-d-1}{n-1} - \binom{-2-d}{n-1} = \binom{-2-d}{n-2}
$$

On the other hand,

$$
\begin{aligned}
\dim_K H^{n-1}(Y, \mathcal{O}_Y(d)) &= \dim_K H^0(Y, \mathcal{O}_Y(-(n-1)-d-1)) = \dim_K H^0(Y, \mathcal{O}_Y(-n-d)) \\
&= \binom{-n-d+(n-1)-1}{(n-1)-1} = \binom{-d-2}{n-2}
\end{aligned}
$$

so we must have that $\dim_K(\ker\beta) = 0$ whence β is injective. Similarly,

$$
\begin{aligned}
\dim_K(\ker\alpha) = \dim_K(\operatorname{im}\delta) &= \dim_K H^0(Y, \mathcal{O}_Y(d)) - \dim_K(\ker\delta) \\
&= \dim_K H^0(Y, \mathcal{O}_Y(d)) - \dim_K(\operatorname{im} f_2) \\
&= \dim_K H^0(Y, \mathcal{O}_Y(d)) - \dim_K H^0(X, \mathcal{O}_X(d)) + \dim_K(\ker f_2) \\
&= \dim_K H^0(Y, \mathcal{O}_Y(d)) - \dim_K H^0(X, \mathcal{O}_X(d)) + \dim_K(\operatorname{im} f_1) \\
&= \dim_K H^0(Y, \mathcal{O}_Y(d)) - \dim_K H^0(X, \mathcal{O}_X(d)) + \dim_K H^0(X, \mathcal{O}_X(d-1)) \\
&= \binom{n-2+d}{n-2} - \binom{n-1+d}{n-1} + \binom{n+d-1}{n} \\
&= 0
\end{aligned}
$$

so that α is injective and δ is the zero map. Now, by induction n, we see that $H^p(Y, \mathcal{O}_Y(d)) = 0$ for all $0 < p < n-1$ whence the maps $H^p(X, \mathcal{O}_X(d-1)) \xrightarrow{\theta_p} H^p(X, \mathcal{O}_X(d))$ are isomorphisms for $0 < p < n$.

Now, using Čech cohomology, the maps β_p are induced by the maps

$$
S(d-1)_{(t_{i_0}\cdots t_{i_p})} = \mathcal{O}_X(d-1)(U_{i_0,\ldots,i_p}) \to \mathcal{O}_X(d)(U_{i_0,\ldots,i_p}) = S(d)_{(t_{i_0}\cdots t_{i_p})}
$$

which is just multiplication by t_n. Hence θ_p is just multiplication by t_n. Now let $\mathcal{F} = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_X(d)$. Then

$$\mathcal{F}(U_{i_0,\dots,i_p}) = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_X(d)(U_{i_0,\dots,i_p}) \cong \bigoplus_{d\in\mathbb{Z}} S(d)_{t_{i_0}\dots t_{i_p}} \cong S_{t_{i_0}\dots t_{i_p}}$$

$$\sum_{d\in\mathbb{Z}} \lambda_d \leftarrow\!\shortmid (\lambda_d)$$

The Čech complex is then

$$0 \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \dots$$

which is nothing but

$$0 \longrightarrow \prod S_{t_{i_0}} \longrightarrow \prod S_{t_{i_0}t_{i_1}} \longrightarrow \dots$$

Localising this complex at t_n gives

$$0 \longrightarrow \prod S_{t_{i_0}t_n} \longrightarrow \prod S_{t_{i_0}t_{i_1}t_n} \longrightarrow \dots$$

But this is the Cech complex of $\mathcal{F}|_{U_n}$ with respect to the cover $\mathcal{U}' = \{ U_i \cap U_n \}_{i\in I}$. But U_n is affine and so $\mathcal{F}|_{U_n}$ is quasi-coherent and so

$$\check{H}^p(\mathcal{U}', \mathcal{F}|_{U_n}) = H^p(U_n, \mathcal{F}|_{U_n}) = 0$$

for all $p > 0$. Hence $H^p(X, \mathcal{F})|_{t_n} = 0$ for all $0 < p < n$. But this means that for all $w \in H^p(X, \mathcal{F})$, there exists r such that $t_n^r w = 0$ which implies that for all $u \in H^p(X, \mathcal{O}_X(d))$, there exists s such that $t_n^s u = 0$. Now, β_p was shown to be multiplication by t_n and we have shown that multiplication by t_n eventually kills every element of $H^p(X, \mathcal{O}_X(d-1))$. Hence, in order for β_p to be an isomorphism, we must have that $H^p(X, \mathcal{O}_X(d)) = 0$ for all $0 < p < n$. □

Proposition 5.0.2. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ a quasi-coherent sheaf on X. Then there exists $l, m \in \mathbb{Z}$ and a surjective homomorphism*

$$\varphi : \bigoplus_{i=1}^{l} \mathcal{O}_X \to \mathcal{F}(m)$$

Proof. Proof omitted. □

Theorem 5.0.3. *Let K be a field and X a closed subscheme of $\mathbb{P}^n_K$ and $f : X \to \mathbb{P}^n_K$ the corresponding closed immersion. If $\mathcal{F}$ is a quasi-coherent sheaf on X then*

$$H^p(X, \mathcal{F}(d)) = 0$$

for all $p > 0$ and for sufficiently $d \in \mathbb{Z}$.

Proof. By definition, we have

$$f_*(\mathcal{F}(d)) \cong (f_*\mathcal{F})(d) = (f_*\mathcal{F}) \otimes_{\mathcal{O}_{\mathbb{P}^n_K}} \mathcal{O}_{\mathbb{P}^n_K}(d)$$

Moreover,

$$H^p(X, \mathcal{F}(d)) \cong H^p(\mathbb{P}^n_K, (f_*\mathcal{F})(d))$$

so we can replace X with $\mathbb{P}^n_K$ and $\mathcal{F}$ with $f_*\mathcal{F}$ and so we can assume that $X = \mathbb{P}^n_K$. Now choose, $l, m \in \mathbb{Z}$ so that we have a surjective homomorphism

$$\varphi : \bigoplus_{i=1}^{l} \mathcal{O}_X \to \mathcal{F}(m)$$

Let $\mathcal{G}$ be the kernel of this morphism so that we have an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_{i=1}^{l} \mathcal{O}_X \longrightarrow \mathcal{F}(m) \longrightarrow 0$$

Tensoring with $\mathcal{O}_X(d-m)$ yields

$$0 \longrightarrow \mathcal{G}(d-m) \longrightarrow \bigoplus_{i=1}^{l} \mathcal{O}_X(d-m) \longrightarrow \mathcal{F}(d) \longrightarrow 0$$

Taking cohomology groups yields a long exact sequence

$$\ldots \longrightarrow H^p(X, \mathcal{G}(d-m)) \longrightarrow H^p\left(X, \bigoplus_{i=1}^{l} \mathcal{O}_X(d-m)\right) \longrightarrow H^n(X, \mathcal{F}(d)) \longrightarrow H^{p+1}(X, \mathcal{G}(d-m)) \longrightarrow \ldots$$

By Theorem 5.0.1, $H^p\left(\bigoplus_{i=1}^{l} \mathcal{O}_X(d-m)\right) = 0$ for all $p > 0$ and large enough $d \in \mathbb{Z}$. By reverse induction, $H^{p+1}(X, \mathcal{G}(d-m)) = 0$ for all $p+1 > n$ since, using Čech cohomology, there are not enough open sets to intersect for $p+1 > n$. This then forces $H^n(X, \mathcal{F}(d)) = 0$ for large enough d and so by induction on p we have $H^p(X, \mathcal{F}(d)) = 0$ for all $p > 0$ and large enough d. $\square$

Theorem 5.0.4. *Let K be a field and X a closed subscheme of $\mathbb{P}^n_K$ and $f : X \to \mathbb{P}^n_K$ the corresponding closed immersion. If $\mathcal{F}$ is a quasi-coherent sheaf on X then $H^p(X, \mathcal{F})$ is a finite-dimensional K-vector space for all p.*

Proof. As before, we can assume that $X = \mathbb{P}^n_K$. Let $m, l \in \mathbb{Z}$ be such that we have an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_{i=1}^{l} \mathcal{O}_X \longrightarrow \mathcal{F}(m) \longrightarrow 0$$

Tensoring with $\mathcal{O}_X(-m)$ yields

$$0 \longrightarrow \mathcal{G}(-m) \longrightarrow \bigoplus_{i=1}^{l} \mathcal{O}_X(-m) \longrightarrow \mathcal{F}(d) \longrightarrow 0$$

Taking cohomology groups yields a long exact sequence

$$\ldots \longrightarrow H^p(X, \mathcal{G}(d-m)) \longrightarrow H^p\left(X, \bigoplus_{i=1}^{l} \mathcal{O}_X(-m)\right) \longrightarrow H^n(X, \mathcal{F}(d)) \longrightarrow H^{p+1}(X, \mathcal{G}(-m)) \longrightarrow \ldots$$

By revere induction on p, we see that for all $p+1 > n$ we have $H^p(X, \mathcal{G}(-m)) = 0$. By Theorem 5.0.1, we know that

$$\dim_K H^p \left(X, \bigoplus_{i=1}^{l} \mathcal{O}_X(-m) \right) < \infty$$

for all p. This implies that $\dim_K H^n(X, \mathcal{F}) < \infty$. By induction on p, we then have that $\dim_K H^p(X, \mathcal{F}) < \infty$. $\square$

5.1 Euler Characteristic and Hilbert Polynomials

Definition 5.1.1. Let K be a field and X a scheme projective over K so that we have a closed immersion $f : X \to \mathbb{P}^n_K$. Let $\mathcal{F}$ be a coherent sheaf over X. We define the **Euler characteristic** of $\mathcal{F}$ to be

$$\chi(X, \mathcal{F}) = \sum_p (-1)^p \dim_K H^p(X, \mathcal{F})$$

Lemma 5.1.2. *Let K be a field and X a scheme projective over K. Suppose that we have an exact sequence of coherent sheaves over X*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \dots \longrightarrow \mathcal{F}_r \longrightarrow 0$$

Then

$$\sum_{i=0}^{r} (-1)^i \chi(X, \mathcal{F}_i) = 0$$

Proof. If $r \leq 2$ then the Lemma is trivial. Now suppose that $r = 3$. Then we have a long exact sequence of cohomology groups

$$\begin{array}{l}
0 \longrightarrow H^0(X, \mathcal{F}_1) \longrightarrow H^0(X, \mathcal{F}_2) \longrightarrow H^0(X, \mathcal{F}_3) \longrightarrow \\
\longrightarrow H^1(X, \mathcal{F}_1) \longrightarrow H^1(X, \mathcal{F}_2) \longrightarrow H^1(X, \mathcal{F}_3) \dashrightarrow \\
\qquad \vdots \\
\dashrightarrow H^{n-1}(X, \mathcal{F}_1) \longrightarrow H^{n-1}(X, \mathcal{F}_2) \longrightarrow H^{n-1}(X, \mathcal{F}_3) \longrightarrow \\
\longrightarrow H^n(X, \mathcal{F}_1) \longrightarrow H^n(X, \mathcal{F}_2) \longrightarrow H^n(X, \mathcal{F}_3) \longrightarrow 0
\end{array}$$

By the Rank-Nullity Theorem, it follows that

$$\dim_K H^0(X, \mathcal{F}_1) - \dim_K H^0(X, \mathcal{F}_2) + \dots = 0$$

Now suppose that $r > 3$. Let $\mathcal{G}$ be the image of $\mathcal{F}_1 \to \mathcal{F}_2$. Then we have exact sequences

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{G} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}_3 \longrightarrow \mathcal{F}_4 \longrightarrow \dots$$

Then by induction we have $\chi(X,\mathcal{F}_1)-\chi(X,\mathcal{F}_2)+\chi(X,\mathcal{G})=0$ and $\chi(X,\mathcal{G})-\chi(X,\mathcal{F}_3)+\cdots=0$. Subtracting these two equations gives us the Lemma. □

Definition 5.1.3. Let K be a field and X a scheme projective over K so that we have a closed immersion $f: X \to \mathbb{P}^n_K$. Let $\mathcal{F}$ be a coherent sheaf over X. We define the **Hilbert polynomial** of $\mathcal{F}$ to be the function

$$\begin{aligned} \phi_{\mathcal{F}} : \mathbb{Z} &\to \mathbb{Z} \\ d &\mapsto \chi(X, \mathcal{F}(d)) \end{aligned}$$

Theorem 5.1.4. *Let K be a field and X a scheme projective over K so that we have a closed immersion $f: X \to \mathbb{P}^n_K$. Let $\mathcal{F}$ be a coherent sheaf over X. Then $\phi_{\mathcal{F}} \in \mathbb{Q}[d]$.*

Proof. Proof omitted (see handwritten notes). □

Example 5.1.5. Let K be a field and $X = \mathbb{P}^n_K$. We shall calculate $\phi_{\mathcal{O}_X}$. We have that

$$\phi_{\mathcal{O}_X}(d) = \chi(X, \mathcal{O}_X(d)) = \dim_K H^0(X, \mathcal{O}_X(d)) - \dim_K H^1(X, \mathcal{O}_X(d)) + \cdots = \dim_K H^0(X, \mathcal{O}_X(d))$$

for large enough d. So we have

$$\phi_{\mathcal{O}_X}(d) = \binom{n+d}{d}$$

for all d.

Example 5.1.6. Let X be a closed subscheme of $\mathbb{P}^n_K$ where K is a field, defined by $\langle h \rangle$ where h is homogeneous of degree r. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_K}(-r) \longrightarrow \mathcal{O}_{\mathbb{P}^n_K} \longrightarrow f_*\mathcal{O}_X \longrightarrow 0$$

so we have

$$\phi_{\mathcal{O}_X}(d) = \phi_{f_*\mathcal{O}_X}(d) = \phi_{\mathcal{O}_{\mathbb{P}^n_K}}(d) - \phi_{\mathcal{O}_{\mathbb{P}^n_K}}(d-r) = \binom{d+n}{d} - \binom{d-r+n}{d-r}$$